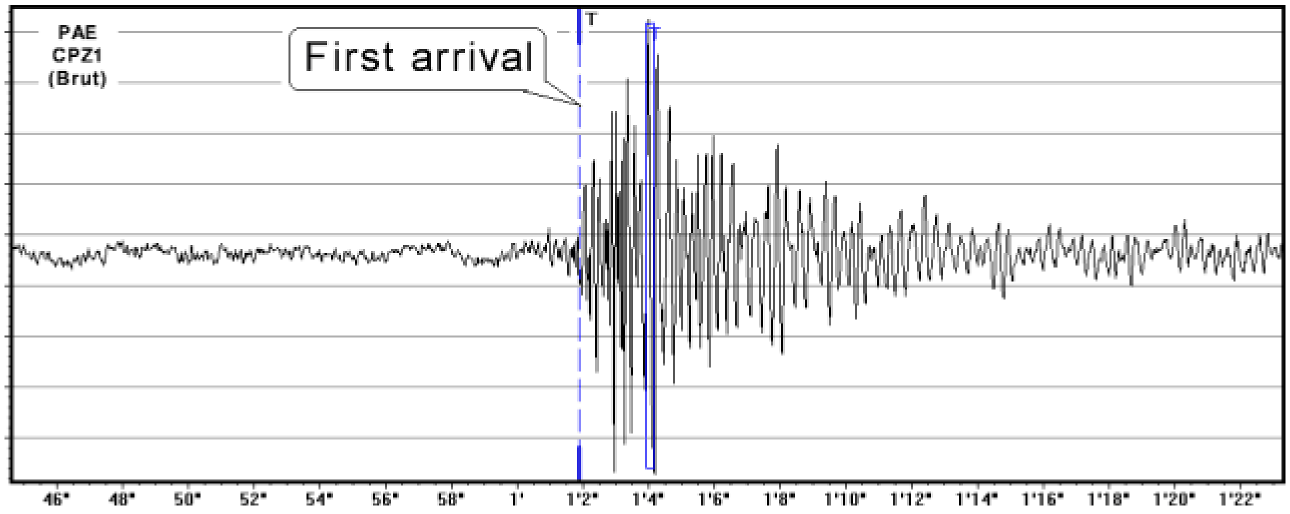
Haar Wavelet Analysis

Why Wavelets?

Wavelets were first applied in geophysics to analyze data from seismic surveys, which are used in oil and mineral exploration to get “pictures” of layering in subsurface rock.
In fact, mathematicians had developed them to solve some abstract problems some twenty years earlier but had not anticipated their applications in signal processing.

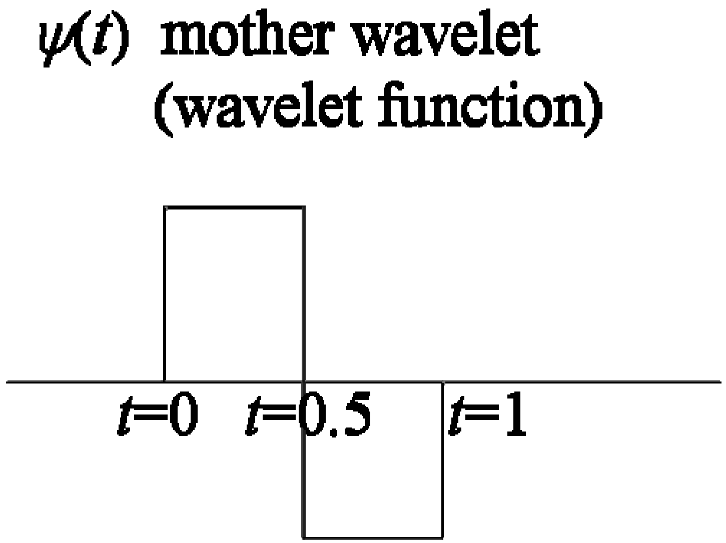


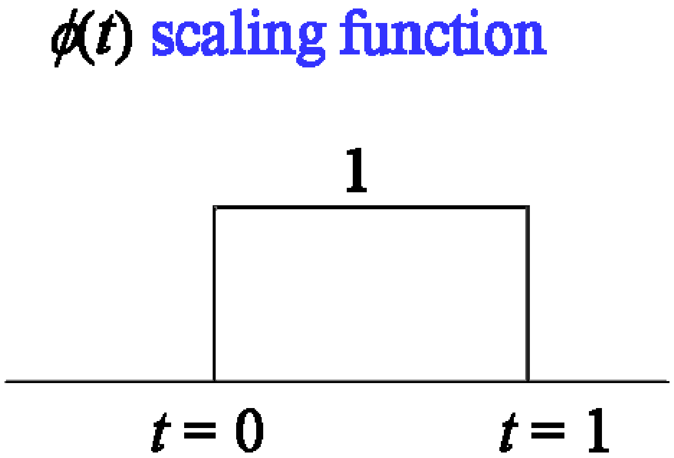
Haar Wavelets

Wavelet can keep track of time and frequency information.

There are two functions that play a primary role in wavelet analysis, the scaling function ϕ (father wavelet) and the wavelet ψ (mother wavelet).

The simplest wavelet analysis is based on Haar scaling function.





Haar Wavelets

The Haar scaling function is defined as

$$\phi(x) = \begin{cases} 1, & \text{if } 0 \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

The Haar Wavelet's mother function is defined as $\psi(x) = \phi(2x) - \phi(2x-1)$

$$\psi(x) = \begin{cases} 1, & 0 \le x < 1/2, \\ -1, & 1/2 \le x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Haar Wavelet's properties:

(1) Any function can be the linear combination of

$\phi(x), \phi(2x), \phi(2^2 x), \cdots \phi(2^k x), \cdots$ and their shifting functions

(2) Any function can be the linear combination of constant function,

$\psi(x), \psi(2x), \psi(2^2 x), \cdots, \psi(2^k x), \cdots$ and their shifting functions

Haar Wavelets

(3) The set of functions $\{2^{j/2}\phi(2^j x - k); k \in Z\}$ is an orthonormal basis.

Proof:

$$\int_{-\infty}^{\infty} (\phi(x))^2 dx = \int_0^1 1 dx = 1$$ j=0

$$\int_{-\infty}^{\infty} (\sqrt{2}\phi(2x))^2 dx = \int_0^{1/2} (\sqrt{2} \cdot 1)^2 dx = 1$$ j=1

$$\int_{-\infty}^{\infty} (2^{j/2}\phi(2^j x))^2 dx = \int_0^{1/2^j} (2^{j/2} \cdot 1)^2 dx = 1$$

for any integer j

k is integer and it is only a translation, it does not affect the integration.

1-level Haar Transform

How to do Haar transform:

Assumption: 1D signal f of the length $N = 2^n$

1-level Haar-Transform for $f = (x_1, x_2, \ldots, x_N)$

$$f \xrightarrow{H_1} (a^1 \mid d^1)$$

where

$$a^1 = \left(\frac{x_1 + x_2}{\sqrt{2}}, \frac{x_3 + x_4}{\sqrt{2}}, \cdots, \frac{x_{N-1} + x_N}{\sqrt{2}}\right)$$

$$d^1 = \left(\frac{x_1 - x_2}{\sqrt{2}}, \frac{x_3 - x_4}{\sqrt{2}}, \cdots, \frac{x_{N-1} - x_N}{\sqrt{2}}\right)$$

1-level Haar Transform

Example:

$$f = (9,7,3,5)$$

$$a^1 = (\frac{16}{\sqrt{2}}, \frac{8}{\sqrt{2}})$$

$$d^1 = (\frac{2}{\sqrt{2}}, \frac{-2}{\sqrt{2}})$$

The transformation H_1 is reversible. That means, f can be reconstructed via (a^1, d^1)

$$a^1 = (a_1, \ldots, a_{N/2})$$

$$d^1 = (d_1, \ldots, d_{N/2})$$

$$f = (\frac{a_1 + d_1}{\sqrt{2}}, \frac{a_1 - d_1}{\sqrt{2}}, \ldots, \frac{a_{N/2} + d_{N/2}}{\sqrt{2}}, \frac{a_{N/2} - d_{N/2}}{\sqrt{2}})$$

1-level Haar Transform

The energy after Haar transform is not changed.

The energy of a signal $f = (x_1, x_2, \ldots, x_N)$ is defined as:

$$E_f = x_1^2 + x_2^2 + \cdots + x_N^2$$

The energy of $(a^1 \mid d^1)$ is

$$a_1^2 + \cdots + a_{N/2}^2 + d_1^2 + \cdots + d_{N/2}^2$$
$$= \frac{(x_1 + x_2)^2}{2} + \frac{(x_1 - x_2)^2}{2} + \cdots + \frac{(x_{N-1} + x_N)^2}{2} + \frac{(x_{N-1} - x_N)^2}{2}$$
$$= x_1^2 + x_2^2 + \cdots + x_{N-1}^2 + x_N^2$$
$$= E_f$$

1-level Haar Transform

Energy distribution

$$\text{Example}: \quad f=(9,7,3,5),\ a^1=(\tfrac{16}{\sqrt{2}},\tfrac{8}{\sqrt{2}}),\ d^1=(\tfrac{2}{\sqrt{2}},\tfrac{-2}{\sqrt{2}})$$

$$E_{a^1}=(\tfrac{16}{\sqrt{2}})^2+(\tfrac{8}{\sqrt{2}})^2=160$$

$$E_{d^1}=(\tfrac{2}{\sqrt{2}})^2+(\tfrac{-2}{\sqrt{2}})^2=4$$

$$E_{a^1}+E_{d^1}=E_f=164$$

$$\text{Energy from } a^1: \tfrac{160}{164}=97.6\%$$

Multilevel Haar Transform

2 – level Haar transform will be defined as :

$$f \xrightarrow{H_2} (a^2, d^2, d^1)$$

where

$$f \xrightarrow{H_1} (a^1, d^1)$$

$$a^1 \xrightarrow{H_1} (a^2, d^2)$$

Example: $f = (9, 7, 3, 5),\ a^1 = (\frac{16}{\sqrt{2}}, \frac{8}{\sqrt{2}}),\ d^1 = (\frac{2}{\sqrt{2}}, \frac{-2}{\sqrt{2}})$

$$a^2 = 12$$

$$d^2 = 4$$

$$f \xrightarrow{H_2} (12 \mid 4 \mid \tfrac{2}{\sqrt{2}} \mid \tfrac{-2}{\sqrt{2}})$$

Multilevel Haar Transform

Example: $f = (4, 6, 10, 12, 8, 6, 5, 5)$

$$a^1 = (\frac{10}{\sqrt{2}}, \frac{22}{\sqrt{2}}, \frac{14}{\sqrt{2}}, \frac{10}{\sqrt{2}}), \quad d^1 = (\frac{-2}{\sqrt{2}}, \frac{-2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 0)$$

$$a^2 = (16, 12), \quad d^2 = (-6, 2)$$

$$a^3 = \frac{28}{\sqrt{2}}, \quad d^3 = \frac{4}{\sqrt{2}}$$

That means

$$f \xrightarrow{H_3} (a^3 \mid d^3 \mid d^2 \mid d^1)$$
$$= (\frac{28}{\sqrt{2}}, \frac{4}{\sqrt{2}}, -6, 2, \frac{-2}{\sqrt{2}}, \frac{-2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 0)$$

Haar Wavelets

$1-\text{level Haar wavelets}:$

$$\mathbf{W}_1^1 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0, \cdots, 0)$$

$$\mathbf{W}_2^1 = (0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \cdots, 0)$$

$$\vdots$$

$$\mathbf{W}_{N/2}^1 = (0, 0, 0, 0, \cdots, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$$

Therefore, d^1 can be represented as:

$$d^1 = (f\mathbf{W}_1^1, f\mathbf{W}_2^1, \ldots, f\mathbf{W}_{N/2}^1)$$

Haar Wavelets

$1-\text{level Haar scaling functions:}$

$$\mathbf{V}_1^1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, \cdots, 0)$$

$$\mathbf{V}_2^1 = (0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \cdots, 0)$$

$$\vdots$$

$$\mathbf{V}_{N/2}^1 = (0, 0, 0, 0, \cdots, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

Therefore, a^1 can be represented as:

$$a^1 = (f\mathbf{V}_1^1, f\mathbf{V}_2^1, \ldots, f\mathbf{V}_{N/2}^1)$$

Haar Wavelets

Reconstruction from 1-level Haar transform

$$
\begin{aligned}
f &= \left(\frac{a_1 + d_1}{\sqrt{2}}, \frac{a_1 - d_1}{\sqrt{2}}, \cdots, \frac{a_{N/2} + d_{N/2}}{\sqrt{2}}, \frac{a_{N/2} - d_{N/2}}{\sqrt{2}}\right) \\
&= \left(\frac{a_1}{\sqrt{2}}, \frac{a_1}{\sqrt{2}}, \cdots, \frac{a_{N/2}}{\sqrt{2}}, \frac{a_{N/2}}{\sqrt{2}}\right) + \left(\frac{d_1}{\sqrt{2}}, -\frac{d_1}{\sqrt{2}}, \cdots, \frac{d_{N/2}}{\sqrt{2}}, -\frac{d_{N/2}}{\sqrt{2}}\right) \\
&= (a_1 \mathbf{V}_1^1 + \cdots + a_{N/2} \mathbf{V}_{N/2}^1) + (d_1 \mathbf{W}_1^1 + \cdots + d_{N/2} \mathbf{W}_{N/2}^1) \\
&= \underbrace{(f \mathbf{V}_1^1)\mathbf{V}_1^1 + \cdots + (f \mathbf{V}_{N/2}^1)\mathbf{V}_{N/2}^1}_{A^1} + \underbrace{(f \mathbf{W}_1^1)\mathbf{W}_1^1 + \cdots + (f \mathbf{W}_{N/2}^1)\mathbf{W}_{N/2}^1}_{D^1} \\
&= A^1 + D^1
\end{aligned}
$$

Haar Wavelets

$\mathbf{V}_1^1, \mathbf{V}_2^1, \cdots, \mathbf{V}_{N/2}^1, \mathbf{W}_1^1, \mathbf{W}_2^1, \cdots, \mathbf{W}_{N/2}^1$ construct an orthonormal basis in an N-dimensional space

$$\mathbf{V}_i^1 \cdot \mathbf{V}_j^1 = 0,\ \ \mathbf{W}_i^1 \cdot \mathbf{W}_j^1 = 0,\ \ i \neq j; \quad \mathbf{V}_i^1 \cdot \mathbf{W}_i^1 = 0$$

$$|\mathbf{V}_i^1| = |\mathbf{W}_i^1| = 1$$

They form a new coordinate system.

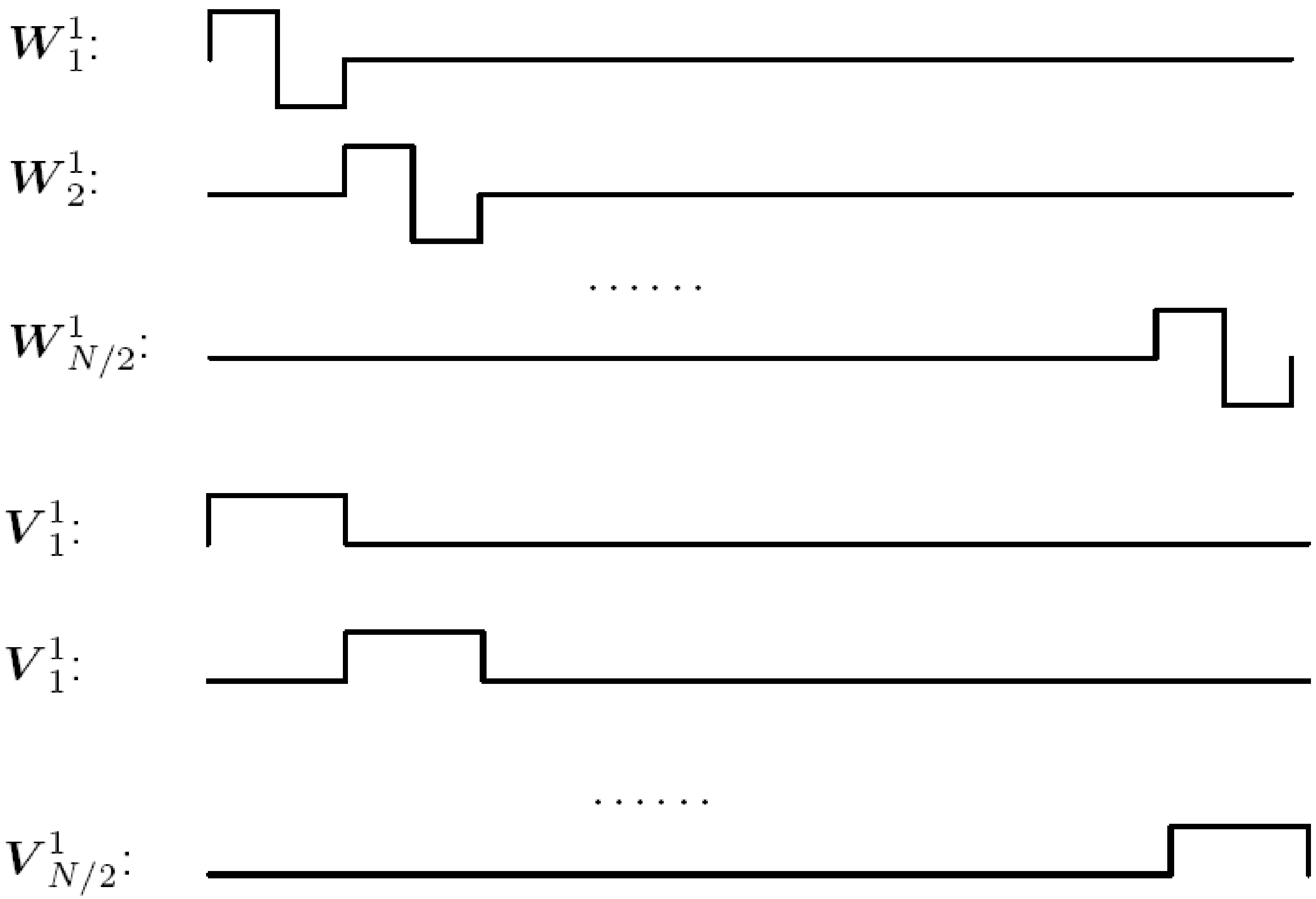
Example: $f = (4,6,10,12,8,6,5,5)$

$$a^1 = \left(\frac{10}{\sqrt{2}}, \frac{22}{\sqrt{2}}, \frac{14}{\sqrt{2}}, \frac{10}{\sqrt{2}}\right),\ \ d^1 = \left(\frac{-2}{\sqrt{2}}, \frac{-2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 0\right)$$

Therefore,

$$f = \frac{10}{\sqrt{2}}\mathbf{V}_1^1 + \frac{22}{\sqrt{2}}\mathbf{V}_2^1 + \frac{14}{\sqrt{2}}\mathbf{V}_3^1 + \frac{10}{\sqrt{2}}\mathbf{V}_4^1 - \frac{2}{\sqrt{2}}\mathbf{W}_1^1 - \frac{2}{\sqrt{2}}\mathbf{W}_2^1 + \frac{2}{\sqrt{2}}\mathbf{W}_3^1$$

Haar Wavelets



Haar Wavelets

For 2-level Haar transform: $a^2 = (f\mathbf{V}_1^2, f\mathbf{V}_2^2, \cdots, f\mathbf{V}_{N/4}^2)$

$$\mathbf{V}_1^2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0,0,0,0,\cdots,0,0,0,0)$$

$$\mathbf{V}_2^2 = (0,0,0,0,\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \cdots,0,0,0,0) \quad \cdots \mathbf{V}_{N/4}^2 = (0,0,0,0,0,0,0,0,\cdots,\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$$

and

$$d^2 = (f\mathbf{W}_1^2, f\mathbf{W}_2^2, \cdots, f\mathbf{W}_{N/4}^2)$$

$$\mathbf{W}_1^2 = (\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, 0,0,0,0,\cdots,0,0,0,0)$$

$$\mathbf{W}_2^2 = (0,0,0,0,\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \cdots,0,0,0,0) \quad \cdots$$

$$\mathbf{W}_{N/4}^2 = (0,0,0,0,0,0,0,0,\cdots\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2})$$

Haar Wavelets

Reconstruction from 2-level Haar transform:

$$f \xrightarrow{H_2} (a^2 \mid d^2 \mid d^1)$$

$$\begin{aligned}
f &= A^2 + D^2 + D^1 \\
D^1 &= (f\mathbf{W}_1^1)\mathbf{W}_1^1 + (f\mathbf{W}_2^1)\mathbf{W}_2^1 + \cdots + (f\mathbf{W}_{N/2}^1)\mathbf{W}_{N/2}^1 \\
D^2 &= (f\mathbf{W}_1^2)\mathbf{W}_1^2 + (f\mathbf{W}_2^2)\mathbf{W}_2^2 + \cdots + (f\mathbf{W}_{N/4}^2)\mathbf{W}_{N/4}^2 \\
A^2 &= (f\mathbf{V}_1^2)\mathbf{V}_1^2 + (f\mathbf{V}_2^2)\mathbf{V}_2^2 + \cdots + (f\mathbf{V}_{N/4}^2)\mathbf{V}_{N/4}^2
\end{aligned}$$