Differentiation
微分

之七

以公式法求函數的微分

連續微分及駐點

successive differentiation and stationary points

Part. 1

SUCCESSIVE DIFFERENTIATION

連續微分

舉例

$$y = x^3 + x^2 + x + 1$$

$$\frac{dy}{dx} = \quad 3x^2 + 2x + 1$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \quad 6x + 2$$

$$\frac{d^3 y}{dx^3} = \frac{d}{dx}\left(\frac{d^2 y}{dx^2}\right) =$$

$$\frac{d^4 y}{dx^4} =$$

Example 4.22　三角函數$\sin ax$的導數(derivatives)

$$f(x) = \sin ax$$

$$f'(x) = a \cos ax$$

$$f''(x) = -a^2 \sin ax = -a^2 f(x)$$

$$f^{(n)}(x) = \begin{cases} (-1)^{(n-1)/2} a^n \cos ax & \text{if } n \text{ is odd (奇數)} \\ (-1)^{n/2} a^n \sin ax & \text{if } n \text{ is even (偶數)} \end{cases}$$

Exercises 連續微分

74. Find all the nonzero derivatives of the function $y = 3x^5 + 4x^4 - 3x^3 + x^2 - 2x + 1$.

$$y = 3x^5 + 4x^4 - 3x^3 + x^2 - 2x + 1$$

$$y' = 3 \times 5x^4 + 4 \times 4x^3 - 3 \times 3x^2 + 2x - 2$$

$$= 15x^4 + 16x^3 - 9x^2 + 2x - 2$$

$$y'' = 15 \times 4x^3 + 16 \times 3x^2 - 9 \times 2x + 2$$

$$= 60x^3 + 48x^2 - 18x + 2$$

$$y''' = 60 \times 3x^2 + 48 \times 2x - 18$$

$$= 180x^2 + 96x - 18$$

$$y^{(4)} = 360x + 96$$

$$y^{(5)} = 360$$

Exercises 對數函數的連續微分

75. Find $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}, \dfrac{d^4y}{dx^4}$ for the function $y = \ln x$.

$$y = \ln x$$

$$\frac{dy}{dx} = 1/x$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(x^{-1}) = -x^{-2} = -1/x^2$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}(-x^{-2}) = 2x^{-3} = 2/x^3$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}(2x^{-3}) = -6x^{-4} = -6/x^4$$

Exercises 指數函數的連續微分

76. Find a general formula for the nth derivative of e^{3x}.

$$\frac{d}{dx}e^{3x} = 3e^{3x}, \quad \frac{d^2}{dx^2}e^{3x} = 3^2 e^{3x}, \quad \frac{d^3}{dx^3}e^{3x} = 3^3 e^{3x},$$

$$\frac{d^{(n)}}{dx^{(n)}}e^{3x} = 3^n e^{3x}$$

Exercises cos函數的連續微分

77. Find a general formula for the nth derivative of $\cos 2x$.

$$y = \cos 2x$$

$$y' = -2 \sin 2x$$

$$y'' = -2^2 \cos 2x$$

$$y^{(3)} = +2^3 \sin 2x$$

$$y^{(4)} = +2^4 \cos 2x$$
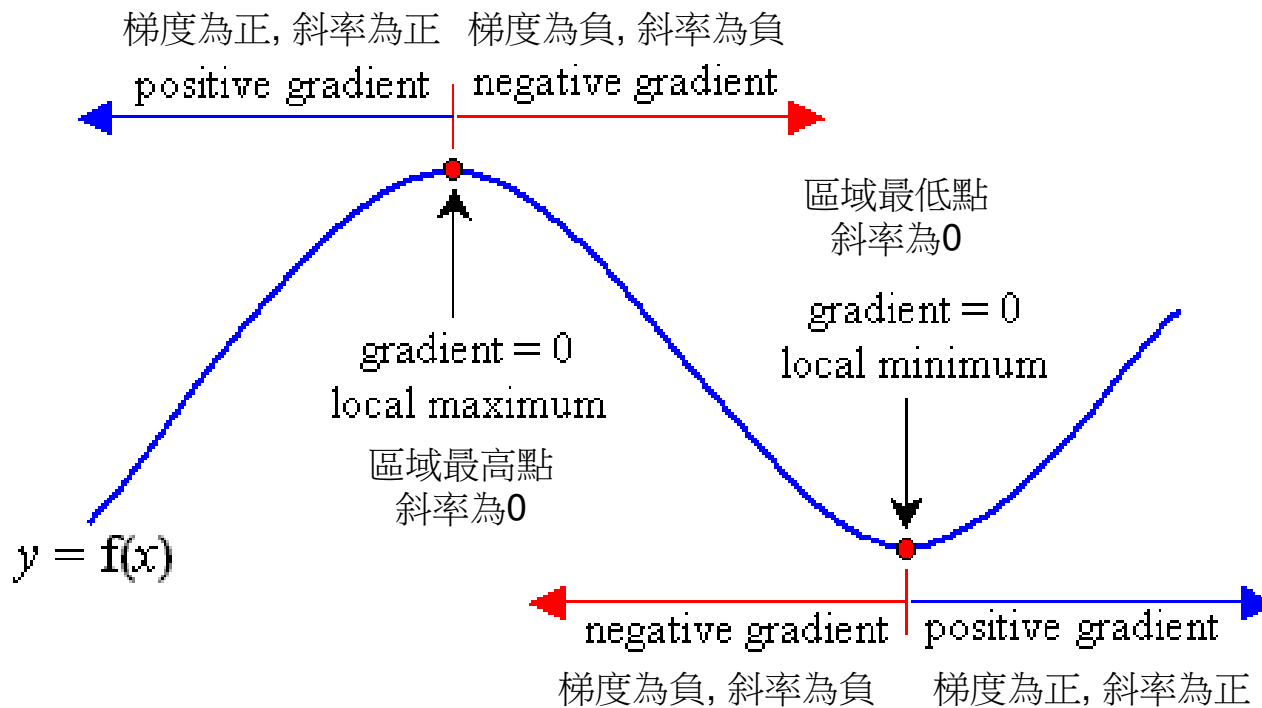
$$y^{(5)} = -2^5 \sin 2x \quad \text{and so on}$$

$$y^{(n)} = (-1)^{n/2} 2^n \cos 2x \qquad \text{when } n \text{ is an even integer (or zero),}$$

$$y^{(n)} = (-1)^{(n+1)/2} 2^n \sin 2x \qquad \text{when } n \text{ is an odd integer.}$$

Part. 2

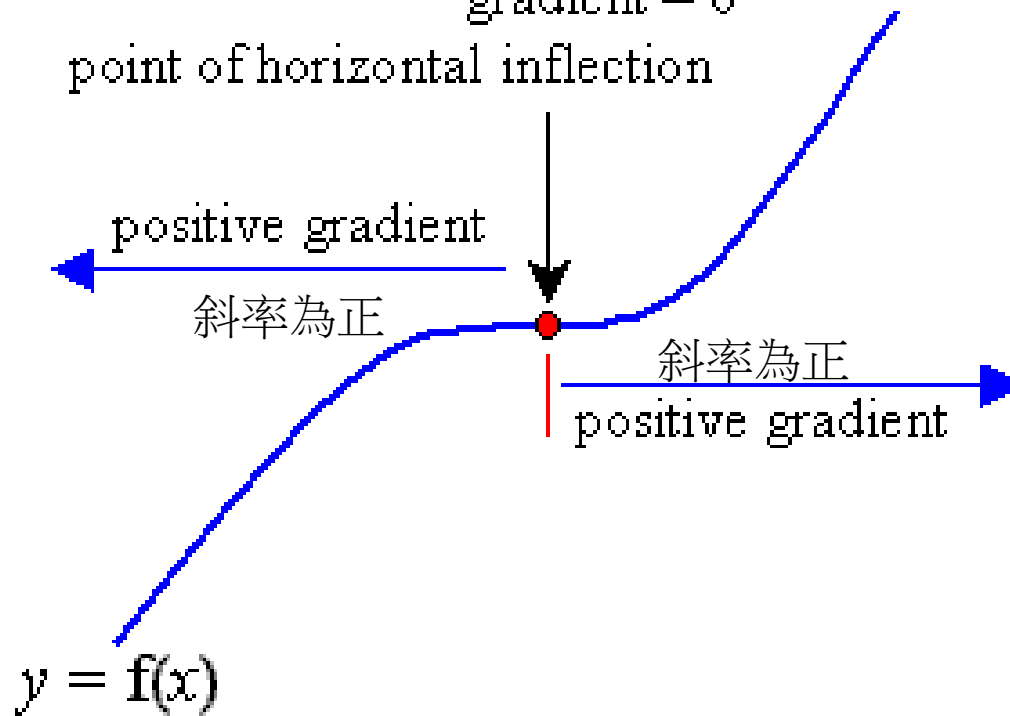# STATIONARY POINT 駐點

Maximum 最高點 與 Minimum 最低點

Inflection point 反曲點

水平反曲點,斜率為0

gradient = 0
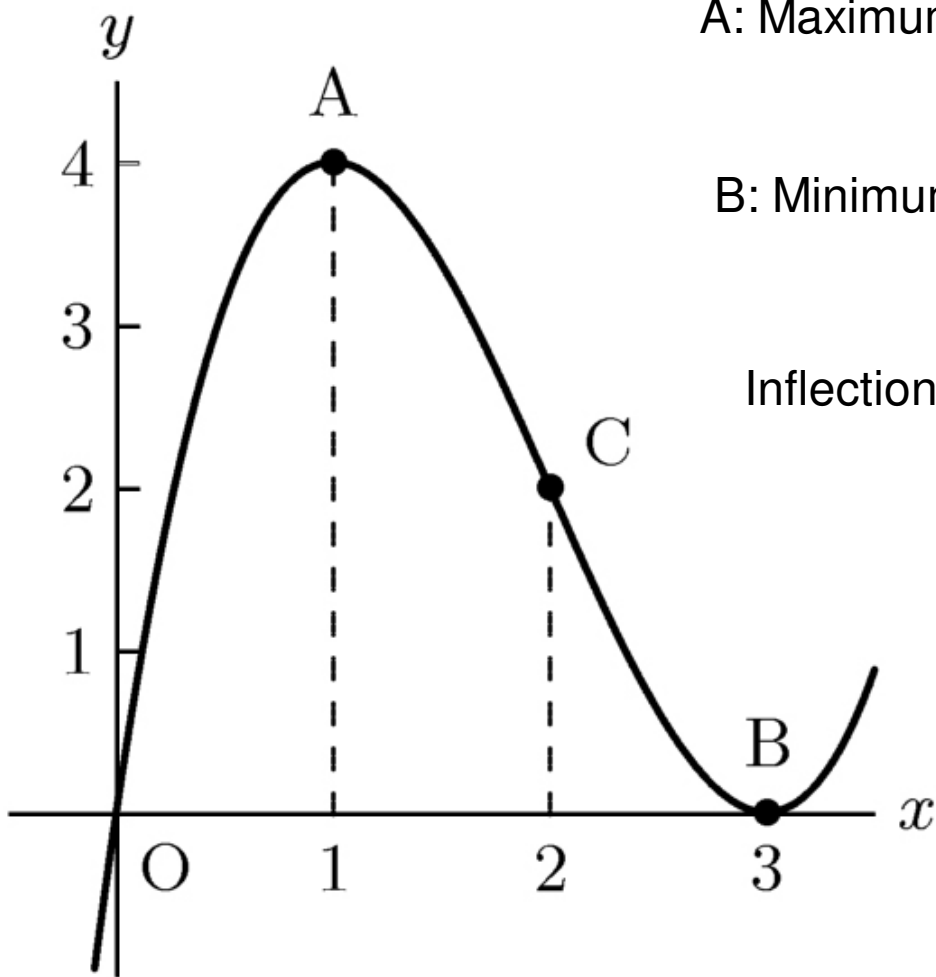
point of horizontal inflection

positive gradient

斜率為正

斜率為正

positive gradient

$y = f(x)$

The first derivative *f'*(*x*) of a function *f*(*x*) is the rate of change of the function, or the slope of its graph at point *x*.

函數 *f*(*x*) 的一次導數 *f'*(*x*) 表示函數變化的速率, 也稱為函數在圖形中某一點*x*的斜率。

The second derivative *f''*(*x*) is the rate of change of slope, and is related to the curvature at *x*.

函數的二次導數 *f''*(*x*) 表示函數協率的變化率, 代表在某一點*x*的曲率(彎曲的程度)。

A: Maximum point $\quad \dfrac{dy}{dx} = 0$ and $\dfrac{d^2 y}{dy^2} < 0$

B: Minimum point $\quad \dfrac{dy}{dx} = 0$ and $\dfrac{d^2 y}{dy^2} > 0$

Inflection point $\quad \dfrac{d^2 y}{dy^2} = 0$

Example 4-23 Find the stationary points 尋找駐點

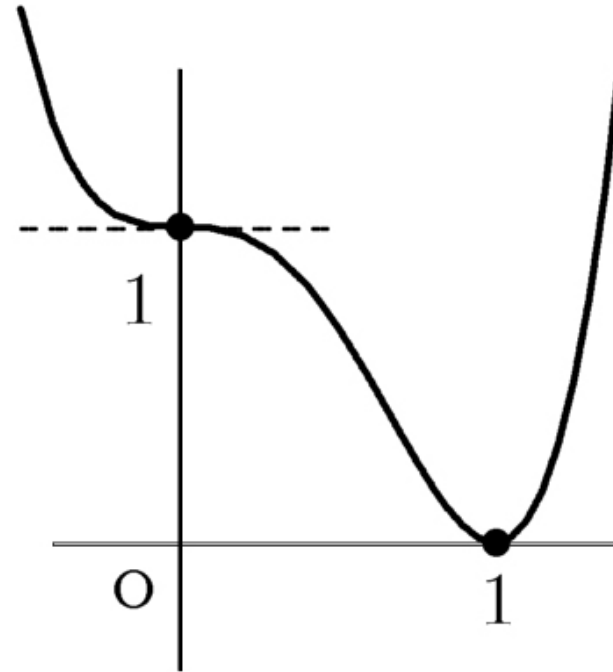$y = 3x^4 - 4x^3 + 1$

$\dfrac{dy}{dx} = 12x^3 - 12x^2 = 12x^2(x-1)$

When $x = 0$ and $x = 1$ $\quad \dfrac{dy}{dx} = 0$

決定駐點的種類：

$\dfrac{d^2y}{dx^2} = 36x^2 - 24x$

$x = 0, \ y = 1 \quad \dfrac{d^2y}{dx^2} = 0; \dfrac{d^3y}{dx^3} = 72x - 24 = -24 \neq 0$ 為反曲點

$x = 1, \ y = 0 \quad \dfrac{d^2y}{dx^2} = 12 > 0$ minimum最低點

Exercise Find the stationary points 尋找駐點

Find the maximum and minimum values and the points of inflection of the following functions.

78. $y = x^2 - 3x + 2$

$$y = x^2 - 3x + 2, \quad \frac{dy}{dx} = 2x - 3$$

when $x = 3/2$ $\quad y = \left(\frac{3}{2}\right)^2 - 3 \times \frac{3}{2} + 3 = -\frac{1}{4}$ $\qquad \frac{dy}{dx} = 0$

$$\frac{d^2 y}{dx^2} = 2 > 0$$

因此 $\left(\frac{3}{2}, -\frac{1}{4}\right)$ 為最低點

Exercise Find the stationary points 尋找駐點

Find the maximum and minimum values and the points of inflection of the following functions.
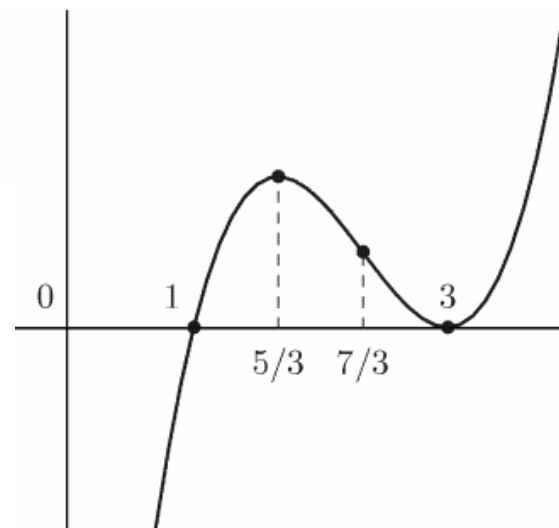
79. $y = x^3 - 7x^2 + 15x - 9$

$y = x^3 - 7x^2 + 15x - 9$

$\dfrac{dy}{dx} = 3x^2 - 14x + 15 = (x-3)(3x-5)$

$\qquad\qquad = 0$ when $x = 3$ and $x = 5/3$

$\dfrac{d^2y}{dx^2} = 6x - 14 \begin{cases} > 0 & \text{when } x = 3, \quad \text{a minimum point} \\ < 0 & \text{when } x = 5/3, \text{ a maximum point} \\ = 0 & \text{when } x = 7/3, \text{ a point of inflection} \end{cases}$

Exercise Find the stationary points 尋找駐點

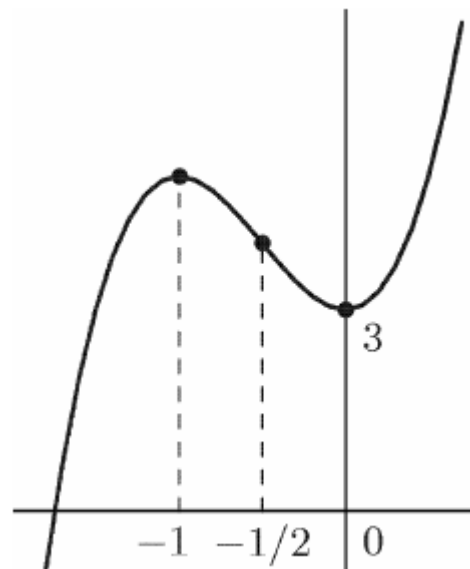Find the maximum and minimum values and the points of inflection of the following functions.

80. $y = 4x^3 + 6x^2 + 3$

$y = 4x^3 + 6x^2 + 3$

$\dfrac{dy}{dx} = 12x^2 + 12x = 12x(x+1)$

$\qquad = 0$ when $x = 0$ and $x = -1$

$\dfrac{d^2y}{dx^2} = 24x + 12 \begin{cases} > 0 & \text{when } x = 0, \ y = 3 & \text{a minimum point} \\ < 0 & \text{when } x = -1, \ y = 5 & \text{a maximum point} \\ = 0 & \text{when } x = -1/2, \ y = 4 & \text{a point of inflection} \end{cases}$
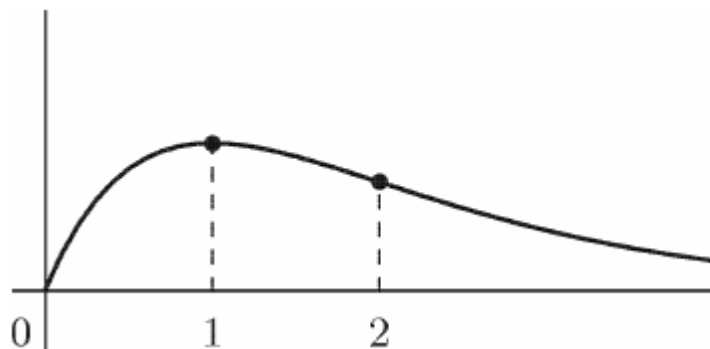
Exercise Find the stationary points 尋找駐點

Find the maximum and minimum values and the points of inflection of the following functions.

81. $y = xe^{-x}$

$y = xe^{-x}$

$$\frac{dy}{dx} = (1-x)e^{-x} = 0 \text{ when } x = 1$$

$$\frac{d^2y}{dx^2} = -(2-x)e^{-x} \begin{cases} < 0 & \text{when } x = 1, \ y = e^{-1} & \text{a maximum point} \\ = 0 & \text{when } x = 2, \ y = 2e^{-2} & \text{a point of inflection} \end{cases}$$

Exercise Find the stationary points 尋找駐點

Find the maximum and minimum values and the points of inflection of the following functions.

82. $y = x^3 - 7x^2 + 16x - 10$

$y = x^3 - 7x^2 + 16x - 10$

$\dfrac{dy}{dx} = 3x^2 - 14x + 16 = (3x - 8)(x - 2)$

$\qquad\qquad = 0$ when $x = 8/3$ and $x = 2$

$\dfrac{d^2y}{dx^2} = 6x - 14 \begin{cases} = +2 > 0 & \text{when } x = 8/3 & \text{a minimum point} \\ = -2 < 0 & \text{when } x = 2 & \text{a maximum point} \end{cases}$