Differentiation

微分

之六

以公式法求函數的微分

對數函數的微分

logarithmic differentiation

Logrithmic function
對數函數

$$y = a^x \qquad \ln y = \ln a^x = x \ln a$$

$$y = u^a v^b w^c \cdots \qquad \ln y = \ln(u^a v^b w^c \cdots)$$

$$= \ln u^a + \ln v^b + \ln w^c \cdots$$

$$= a \ln u + n \ln v + c \ln w \cdots$$

Logrithmic function

對數函數

$$y = u^a v^b w^c \cdots$$

$$\ln y = a \ln u + n \ln v + c \ln w \cdots$$

$$\frac{d \ln y}{dx} = \frac{1}{y}\frac{dy}{dx} = a\frac{d \ln u}{dx} + b\frac{d \ln v}{dx} + c\frac{d \ln w}{dx} + \cdots$$

$$= \frac{a}{u}\frac{du}{dx} + \frac{b}{v}\frac{dv}{dx} + \frac{c}{w}\frac{dw}{dx} + \cdots$$

Example 4.20 對數函數的微分

$$y=\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} \quad \text{Find } \frac{dy}{dx}=?$$

$$y=\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} \qquad \ln y=\ln\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}=\frac{1}{2}\left[\ln(1+x)-\ln(1-x)\right]$$

$$\frac{d\ln y}{dx}=\frac{1}{2}\left[\frac{d\ln(1+x)}{dx}-\ln\frac{d\ln(1-x)}{dx}\right]=\frac{1}{2}\left[\frac{1}{1+x}\frac{d(1+x)}{dx}-\frac{1}{1-x}\frac{d(1-x)}{dx}\right]$$

$$=\frac{1}{2}\left(\frac{1}{1+x}+\frac{1}{1-x}\right)=\frac{1}{2}\times\frac{2}{1-x^2}=\frac{1}{1-x^2}$$

$$\frac{d\ln y}{dx}=\frac{1}{y}\frac{dy}{dx}=\frac{1}{1-x^2}$$

$$\frac{dy}{dx}=\frac{y}{1-x^2}$$

Example 4.20 對數函數的微分

$$y = a^x \quad \text{Find}\ \frac{dy}{dx} = ?$$

$$y = a^x \qquad \ln y = x \ln a$$

$$\frac{d \ln y}{dx} = x \frac{d \ln a}{dx} + \ln a \frac{dx}{dx} = \ln a$$

$$\frac{d \ln y}{dx} = \frac{1}{y} \frac{dy}{dx}$$

$$\frac{1}{y} \frac{dy}{dx} = \ln a \qquad \frac{dy}{dx} = y \ln a = a^x \ln a$$

Example 4.20 對數函數的微分

$$y = x^x \quad \text{Find} \frac{dy}{dx} = ?$$

$$y = x^x \qquad \ln y = x \ln x$$

$$\frac{d \ln y}{dx} = x \frac{d \ln x}{dx} + \ln x \frac{dx}{dx} = x \times \frac{1}{x} + \ln x = 1 + \ln x$$

$$\frac{d \ln y}{dx} = \frac{1}{y} \frac{dy}{dx}$$

$$\frac{1}{y} \frac{dy}{dx} = 1 + \ln x \qquad \frac{dy}{dx} = y(1 + \ln x) = x^x (1 + \ln x)$$

Exercises對數函數的微分

微分下列函數

69. $\left(\dfrac{3-x}{4+x}\right)^{1/3}$

$$y=\left(\frac{3-x}{4+x}\right)^{1/3}\rightarrow \ln y=\frac{1}{3}\left[\ln(3-x)-\ln(4+x)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{3}\left[-\frac{1}{3-x}-\frac{1}{4+x}\right]=-\frac{7}{3(3-x)(4+x)}$$

$$\frac{dy}{dx}=-\frac{7}{3(3-x)(4+x)}\left(\frac{3-x}{4+x}\right)^{1/3}$$

Exercises對數函數的微分

微分下列函數

70. $$\frac{(1+x^2)(x-1)^{1/2}}{(2x+1)(3x^2+2x-1)^{1/3}}$$

$$y=\frac{(1+x^2)(x-1)^{1/2}}{(2x+1)(3x^2+2x-1)^{1/3}}$$

$$\rightarrow \ln y=\ln(1+x^2)+\frac{1}{2}\ln(x-1)-\ln(2x+1)-\frac{1}{3}\ln(3x^2+2x-1)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{2x}{1+x^2}+\frac{1}{2(x-1)}-\frac{2}{2x+1}-\frac{6x+2}{3(3x^2+2x-1)}$$

$$\frac{dy}{dx}=\left[\frac{2x}{1+x^2}+\frac{1}{2(x-1)}-\frac{2}{2x+1}-\frac{6x+2}{3(3x^2+2x-1)}\right]\times\frac{(1+x^2)(x-1)^{1/2}}{(2x+1)(3x^2+2x-1)^{1/3}}$$

Exercises對數函數的微分

微分下列函數

71. $\sin^{1/2} x \cos^3(x^2+1)\tan^{1/3} 2x$

$$y=\sin^{1/2} x \cos^3(x^2+1)\tan^{1/3} 2x$$

$$\to \ln y=\frac{1}{2}\ln(\sin x)+3\ln(\cos(x^2+1))+\frac{1}{3}\ln(\tan 2x)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\frac{1}{\sin x}\times\cos x+\frac{3}{\cos(x^2+1)}\times(-\sin(x^2+1))\times 2x+\frac{1}{3}\frac{1}{\tan 2x}\times\sec^2 2x\times 2$$

$$=\left[\frac{1}{2}\cot x-6x\tan(x^2+1)+\frac{4}{3\sin 4x}\right]$$

$$\frac{dy}{dx}=\left[\frac{1}{2}\cot x-6x\tan(x^2+1)+\frac{4}{3\sin 4x}\right]\times\sin^{1/2} x\cos^3(x^2+1)\tan^{1/3} 2x$$

Exercises 對數函數微分的應用

72. Show that the equations

$$\frac{d\ln p}{dT} = \frac{\Delta H_{\text{vap}}}{RT^2} \quad \text{and} \quad \frac{dp}{dT} = \frac{p\Delta H_{\text{vap}}}{RT^2}$$

are equivalent expressions of the Clausius-Clapeyron equation.

$$\frac{d\ln p}{dT} = \frac{1}{p}\frac{dp}{dT}$$

$$\frac{1}{p}\frac{dp}{dT} = \frac{\Delta H_{\text{vap}}}{RT^2} \quad \text{and} \quad \frac{dp}{dT} = \frac{p\Delta H_{\text{vap}}}{RT^2}$$

Exercises 對數函數微分的應用

73. The decomposition of dinitrogen pentoxide in tetrachloromethane at $T = 45\ ^\circ\mathrm{C}$ has stoichiometry:

$$N_2O_5 \rightarrow 2NO_2 + \tfrac{1}{2}O_2$$

and obeys first-order kinetics. From the volumes of oxygen liberated after various times *t*, the following concentrations of N_2O_5 were obtained:

$x = [N_2O_5]/\mathrm{mol\ dm^{-3}}$	2.33	1.91	1.36	1.11	0.72	0.55
t/s	0	319	867	1196	1877	2315

Plot a graph of $\ln x$ against t/s and determine the rate constant.

Exercises 對數函數微分的應用

一級反應

$$x(t) = x_0 e^{-kt}$$

$$\ln x = \ln x_0 + \ln e^{-kt} = \ln x_0 - kt$$

註解:

x_0: 反應物的起始濃度, 時間t=0的反應物的濃度

$x(t)$: 在時間t時反應物的濃度

k: 速率常數

Exercises 對數函數微分的應用

一級反應

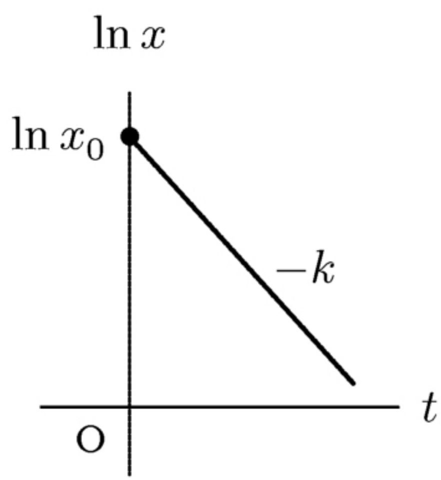
$$\ln x = \ln x_0 - kt$$

線性關係

$y = ax + b$

a : 斜率

b : 截距



註解:

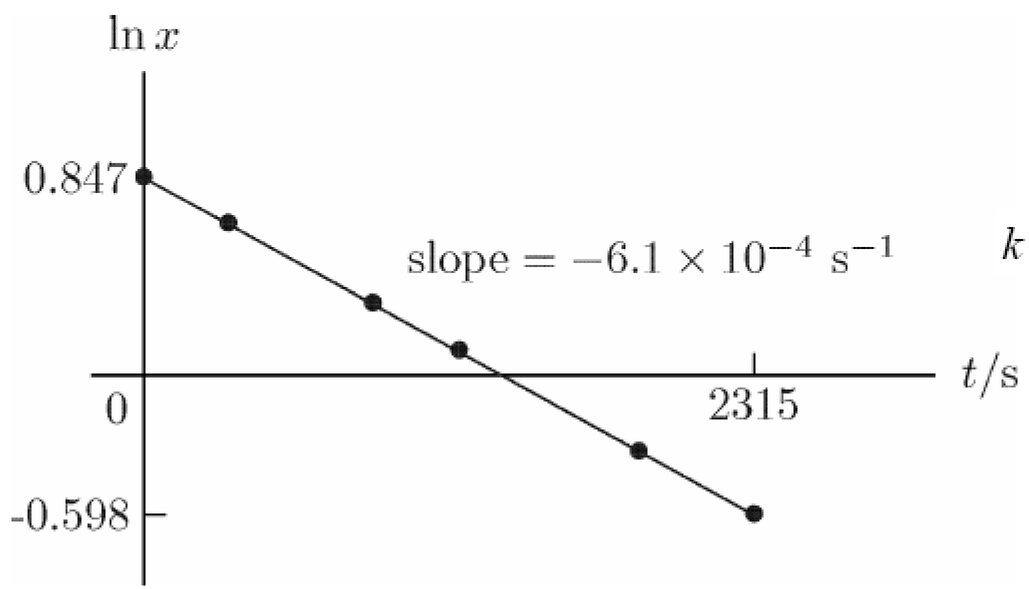
x_0: 反應物的起始濃度, 時間t=0的反應物的濃度

$x(t)$: 在時間t時反應物的濃度

k: 速率常數

$x = [N_2O_5]/\text{mol dm}^{-3}$	2.33	1.91	1.36	1.11	0.72	0.55
t/s	0	319	867	1196	1877	2315

$\ln x$	0.846	0.647	0.307	0.104	–0.329	–0.598
t/s	0	319	867	1196	1877	2315



$k = 6.1 \times 10^{-4}\ \text{s}^{-1}$