Differentiation

微分

之四

以公式法求函數的微分

反函數的微分

Inverse Function

Inverse function

反函數的微分

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Example 4.14

$x = y^5 - 2y$; 求 $dy/dx = ?$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{\left(\frac{d(y^5 - 2y)}{dy}\right)} = \frac{1}{5y^4 - 2}$$

Exercises *differentiate by rule* 由公式求微分

Inverse function 反函數的微分

Question	Answer
56. If $x = 2y^2 - 3y + 1$, find $\dfrac{dy}{dx}$.	$x = 2y^2 - 3y + 1,\quad \dfrac{dx}{dy} = 4y - 3$ $\dfrac{dy}{dx} = 1\Big/\dfrac{dx}{dy} = \dfrac{1}{4y-3}$

Exercises *differentiate by rule* 由公式求微分 *Inverse function* 反函數的微分

- 氣體狀態方程式的微分

  Differentiation of the state function of gas

57. $pV = nRT\left(1 + \frac{nB}{V}\right)$ Find $\frac{dV}{dp}$ (n, T, B are constants)

$$pV = nRT\left(1 + \frac{nB}{V}\right) \rightarrow p = nRT\left(\frac{1}{V} + \frac{nB}{V^2}\right)$$

$$\frac{dp}{dV} = nRT\left(-\frac{1}{V^2} - 2\frac{nB}{V^3}\right) = -\frac{nRT}{V^3}(V + 2nB)$$

$$\frac{dV}{dp} = \frac{-V^3}{nR(V + 2nB)T}$$

Exercises *differentiate by rule* 由公式求微分 *Inverse function* 反函數的微分

- 氣體狀態方程式的微分

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58. $p(V-nb)-nRT=0$ Find $\dfrac{dV}{dp}$ (n, T, *b* are constants)

$$p(V-nb)-nRT=0 \rightarrow p=\frac{nRT}{V-nb}$$

$$\frac{dp}{dV}=-\frac{nRT}{(V-nb)^2}=-\frac{p}{V-nb}$$

$$\frac{dV}{dp}=-\frac{V-nb}{p}$$

Exercises *differentiate by rule* 由公式求微分 *Inverse function* 反函數的微分

- 氣體狀態方程式的微分

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59. $\left(p+\frac{n^2a}{V^2}\right)(V-nb)=nRT$ Find $\frac{dV}{dp}$ (n, T, *a*, *b* are constants)

$$\left(p+\frac{n^2a}{V^2}\right)(V-nb)=nRT \rightarrow p=\frac{nRT}{V-nb}-\frac{n^2a}{V^2}$$

$$\frac{dp}{dV}=-\frac{nRT}{(V-nb)^2}+\frac{2n^2a}{V^3}$$

$$\frac{dV}{dp}=\left[\frac{2n^2a}{V^3}-\frac{nRT}{(V-nb)^2}\right]^{-1}$$

Exercises *differentiate by rule* 由公式求微分
Inverse function 反函數的微分

- 三角函數反函數的微分

  Differentiation of the inverse trigonometric function

$$\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right)=\frac{1}{\sqrt{a^2-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right)=-\frac{1}{\sqrt{a^2-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right)=\frac{a}{a^2+x^2}$$

Example 4.17 證明 $\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right)=\frac{1}{\sqrt{a^2-x^2}}$

If $y=\sin^{-1}\left(\frac{x}{a}\right)$

$x=a\sin y$

$$\frac{dx}{dy}=\frac{d\left(a\sin y\right)}{dy}=a\cos y$$

because $\sin^2 x+\cos^2 x=1$

$$a\cos y=a\sqrt{1-\sin^2 y}=\sqrt{a^2(1-\sin^2 y)}=\sqrt{a^2-a^2\sin^2 y}=\sqrt{a^2-x^2}$$

$$\frac{dy}{dx}=\frac{1}{\left(\frac{dx}{dy}\right)}=\frac{1}{\sqrt{a^2-x^2}}$$

Exercises *differentiate by rule* 由公式求微分

對下列各函數進行微分

60. $\sin^{-1} 2x$

$$y=\sin^{-1}2x \rightarrow x=\frac{1}{2}\sin y$$

$$\frac{dx}{dy}=\frac{1}{2}\cos y \rightarrow \frac{dy}{dx}=\frac{2}{\cos y}=\frac{2}{\sqrt{1-4x^2}}$$

也可以用公式解

$$y=\sin^{-1}2x=\sin^{-1}\frac{x}{\left(\frac{1}{2}\right)};\quad \text{因此} a=\frac{1}{2}$$

$$\text{代公式可得}\frac{dy}{dx}=\frac{d}{dx}\left(\sin^{-1}\frac{x}{(\frac{1}{2})}\right)=\frac{1}{\sqrt{\left(\frac{1}{2}\right)^2-x^2}}=\frac{1}{\sqrt{\frac{1-4x^2}{4}}}=\frac{2}{\sqrt{1-4x^2}}$$

Exercises *differentiate by rule* 由公式求微分

對下列各函數進行微分

61. $\tan^{-1} x^2$

$$y = \tan^{-1} x^2$$

設$u = x^2 \rightarrow y = \tan^{-1} u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{d\left(\tan^{-1} u\right)}{du} \times \frac{dx^2}{dx} = \left(\frac{1}{1^2 + u^2}\right) \times 2x = \frac{2x}{1 + x^4}$$

Exercises *differentiate by rule* 由公式求微分

對下列各函數進行微分

62. $\cos^{-1}\left(\dfrac{1-x}{1+x}\right)$

$$y=\cos^{-1}\left(\frac{1-x}{1+x}\right)$$

設 $u=\left(\dfrac{1-x}{1+x}\right)\rightarrow y=\cos^{-1}u$

$$\frac{dy}{dx}=\frac{d\cos^{-1}u}{du}\times\frac{du}{dx}=\frac{d\left(\cos^{-1}u\right)}{du}\times\frac{d\left(\frac{1-x}{1+x}\right)}{dx}$$

$$\frac{d\left(\cos^{-1}u\right)}{du}=-\frac{1}{\sqrt{1-u^2}}=-\frac{1}{\left(1-\left(\frac{1-x}{1+x}\right)^2\right)^{1/2}}=\frac{1}{\left(\frac{1+2x+x^2-1+2x-x^2}{(1+x)^2}\right)^{1/2}}=\left(\frac{(1+x)^2}{4x}\right)^{1/2}=\frac{1+x}{2\sqrt{x}}$$

$$\frac{d\left(\frac{1-x}{1+x}\right)}{dx}=\frac{d(1-x)(1+x)^{-1}}{dx}=(1-x)\frac{d(1+x)^{-1}}{dx}+(1+x)^{-1}\frac{d(1-x)}{dx}$$

$$=(1-x)\times[-(1+x)^{-2}]+(1+x)^{-1}\times(-1)$$

$$=-\frac{(1-x)}{(1+x)^2}-\frac{1}{(1+x)}=\frac{-1+x-1-x}{(1+x)^2}=\frac{-2}{(1+x)^2}$$

$$\frac{dy}{dx}=\frac{d\left(\cos^{-1}u\right)}{du}\times\frac{d\left(\frac{1-x}{1+x}\right)}{dx}=\frac{(1+x)}{2\sqrt{x}}\times\frac{-2}{(1+x^2)^2}=-\frac{1}{\sqrt{x}(1+x)}$$