Differentiation

微分

之三

以公式法求函數的微分

連鎖律

(chain rule)

Differentiation of Combinations of functions
組合函數的微分法則

Type 函數形式	Rule 法則
Multiple of a function函數的倍數	$\frac{d}{dx}(au) = a\frac{du}{dx}$
Sum of functions 函數相加	$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$
Product rule 乘法律	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
Quotient rule 除法律	$\frac{d}{dx}(\frac{u}{v}) = \left(v\frac{du}{dx} - u\frac{dv}{dx}\right)/v^2$
Chain rule 連鎖律	$\frac{d}{dx}f(u) = \frac{df}{du} \times \frac{du}{dx}$
Inverse rule 倒數律	$\frac{dx}{dy} = 1 \Big/ \left(\frac{dy}{dx}\right)$ or $\frac{dx}{dy} \times \frac{dy}{dx} = 1$

Chain Rule 連鎖律 $\frac{d}{dx}f(u)=\frac{df}{du}\times\frac{du}{dx}$

Differeciate $y=f(x)=(2x^2-1)^3$

設$u=(2x^2-1)$

$f(u)=u^3$

$$\frac{d}{dx}f(u)=\frac{df}{du}\times\frac{du}{dx}=\frac{du^3}{du}\times\frac{d(2x^2-1)}{dx}$$

$$=3u^2\times(4x+0)=3(2x^2-1)\times4x=12x(2x^2-1)$$

Example 4.14

$$\frac{d}{dx}f(u)=\frac{df}{du}\times\frac{du}{dx}$$

Differentiate $y = (2x^2\text{-}1)^{5/2}$

$$設u=(2x^2-1)$$

$$y=f(u)=u^{5/2}$$

$$\frac{d}{dx}f(u)=\frac{df}{du}\times\frac{du}{dx}=\frac{du^{5/2}}{du}\times\frac{d(2x^2-1)}{dx}$$

$$=(5/2)u^{(5/2)-1}\times(4x+0)=10x(2x^2-1)^{3/2}$$

Exercises *differentiate by rule* 由公式求微分

對下列各函數進行微分

37. $(1+x)^5$ **:** $y=u^5$, where $u=(1+x)$ $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}=(5u^4)\times(1)=5(1+x)^4$

38. $\sqrt{2+x^2}$ **:** $y=u^{1/2}$, where $u=(2+x^2)$ $\dfrac{dy}{dx}=\dfrac{1}{2}u^{-1/2}\times(2x)=x(2+x^2)^{-1/2}=\dfrac{x}{\sqrt{2+x^2}}$

39. $\dfrac{1}{3-x^2}$ **:** $y=u^{-1}$, where $u=3-x^2$ $\dfrac{dy}{dx}=-u^{-2}\times(-2x)=2x(3-x^2)^{-2}=\dfrac{2x}{(3-x^2)^2}$

40. $\dfrac{3}{(2x^2-3x-1)^{1/2}}$ **:** $y=3u^{-1/2}$, where $u=(2x^2-3x-1)$

$$\frac{dy}{dx}=-\frac{3}{2}u^{-3/2}\times(4x-3)=-\frac{3}{2}(4x-3)\Big/(2x^2-3x-1)^{3/2}$$

The Chain rule 連鎖律公式

Type 函數形式	Function 函數	Derivative 微分(導數)
Power of u u的次方	u^a	$au^{a-1}\dfrac{du}{dx}$
Exponential 指數函數	e^u	$e^u\dfrac{du}{dx}$
Logarithmic 對數函數	$\ln\ u$	$\dfrac{1}{u}\dfrac{du}{dx}$
Trigonometric 三角函數	$\sin\ u$	$\cos u\dfrac{du}{dx}$
	$\cos\ u$	$-\sin u\dfrac{du}{dx}$
	$\tan\ u$	$\sec u\dfrac{du}{dx}$

Example 4.15

$$\frac{d}{dx}f(u)=\frac{df}{du}\times\frac{du}{dx}$$

完成下列微分

$$y=\sin 2x$$

$$u=2x;\sin 2x=\sin u$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{d\sin u}{du}\times\frac{d(2x)}{dx}=2\cos u=2\cos 2x$$

$$y=\cos(2x^2-1)$$

$$u=2x^2\text{ -}1;\cos(2x^2-1)=\cos u$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{d\cos u}{du}\times\frac{d(2x^2-1)}{dx}=(-\sin u)\times 4x=-4x\sin(2x^2-1)$$

Example 4.15

$$\frac{d}{dx}f(u)=\frac{df}{du}\times\frac{du}{dx}$$

完成下列微分

$$y=e^{2x^2-1}$$

$$u=2x^2\text{ -}1;\ e^{(2x^2-1)}=e^u$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{de^u}{du}\times\frac{d(2x^2-1)}{dx}=e^u\times 4x=4xe^{(2x^2-1)}$$

$$y=\ln(2x^2-1)$$

$$u=2x^2\text{ -}1;\ \ln(2x^2\text{ -}1)=\ln u$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{d\ln u}{du}\times\frac{d(2x^2-1)}{dx}=\frac{1}{u}\times 4x=\frac{4x}{2x^2-1}$$

Example 4.15

$$\frac{d}{dx}f(u)=\frac{df}{du}\times\frac{du}{dx}$$

完成下列微分

$$y=\ln(\sin x)$$

$$u=\sin x;\ln(\sin x)=\ln u$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{d\ln u}{du}\times\frac{d\sin x}{dx}=\frac{1}{u}\times\cos x=\frac{\cos x}{\sin x}$$

Exercises *differentiate by rule* 由公式求微分

對下列各函數進行微分

41. $\sin 4x$ **:** $y = \sin u$, where $u = 4x$ $\dfrac{dy}{dx} = \cos u \times (4) = 4\cos 4x$

42. e^{-2x} **:** $y = e^u$, where $u = -2x$ $\dfrac{dy}{dx} = e^u \times (-2) = -2e^{-2x}$

43. e^{2x^2-3x+1} **:** $y = e^u$, where $u = 2x^2 - 3x + 1$

$$\frac{dy}{dx} = e^u \times (4x-3) = (4x-3)e^{2x^2-3x+1}$$

Exercises *differentiate by rule* 由公式求微分

對下列各函數進行微分

44. $\ln(2x^2-3x+1)$: $y=\ln u,$ where $u=2x^2-3x+1$

$$\frac{dy}{dx}=\frac{1}{u}\times(4x-3)=\frac{(4x-3)}{2x^2-3x+1}$$

45. $\cos(2x^2-3x+1)$: $y=\cos u,$ where $u=2x^2-3x+1$

$$\frac{dy}{dx}=-\sin u\times(4x-3)=-(4x-3)\sin(2x^2-3x+1)$$

Exercises *differentiate by rule* 由公式求微分

對下列各函數進行微分

46. $e^{\sin x}$ **:**

$y = e^u$, where $u = \sin x$

$$\frac{dy}{dx} = e^u \times \cos x = e^{\sin x} \cos x$$

47. $\ln(\cos x)$ **:**

$y = \ln u$, where $u = \cos x$

$$\frac{dy}{dx} = \frac{1}{u} \times (-\sin x) = -\frac{\sin x}{\cos x} = -\tan x$$

Exercises *differentiate by rule* 由公式求微分

對下列各函數進行微分

48. $e^{-\cos(3x^2+2)}$**:** $y = e^u$, where $u = -\cos(3x^2+2)$

$$= -\cos v,\ \text{where}\ v = 3x^2+2$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \text{ and } \frac{du}{dx} = \frac{du}{dv} \times \frac{dv}{dx}.$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$$

$$= (e^u) \times (\sin v) \times (6x)$$

$$= 6x\sin(3x^2+2)e^{-\cos(3x^2+2)}$$

Exercises *differentiate by rule* 由公式求微分

對下列各函數進行微分

49. $\ln\left(\dfrac{2+x}{3-x}\right)$**:**

$y = \ln(2+x) - \ln(3-x) = \ln u - \ln v$, where $u = 2+x$ and $v = 3-x$

$$\frac{dy}{dx} = \frac{1}{u}\times 1 - \frac{1}{v}\times(-1)$$

$$= \frac{1}{2+x} + \frac{1}{3-x}$$

Exercises *differentiate by rule* 由公式求微分

對下列各函數進行微分

50. $\ln(\sin 2x + \sin^2 x)$：

$u = \sin 2x + \sin^2 x \qquad y = \ln u,$

$v = 2x$ and $w = \sin x \qquad u = \sin v + w^2$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{dy}{du} \times \left[\frac{d}{dx} \sin v + \frac{d}{dx} w^2 \right]$$

$$= \frac{1}{u} \left[\cos v \times 2 + 2w \times \cos x \right]$$

$$= \frac{2\cos 2x + 2\sin x \cos x}{\sin 2x + \sin^2 x} = \frac{2\cos 2x + \sin 2x}{\sin 2x + \sin^2 x}$$

Exercises *differentiate by rule* 由公式求微分

對下列各函數進行微分

51. $3x^2(2+x)^{1/2}$ **:**

$$y = 3x^2(2+x)^{1/2} = u \times v$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = 3x^2 \times \frac{1}{2}(2+x)^{-1/2} + (2+x)^{1/2} \times 6x$$

$$= 6x(2+x)^{1/2} + \frac{3x^2}{2}(2+x)^{-1/2}$$

Exercises *differentiate by rule* 由公式求微分

對下列各函數進行微分

52. $\sin x \cos 2x$:

$$y = \sin x \cos 2x = u \times v$$

$$u = \sin x,\ \frac{du}{dx} = \cos x; \quad v = \cos 2x,\ \frac{dv}{dx} = -2\sin 2x$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = \sin x \times (-2\sin 2x) + \cos 2x \times \cos x$$

$$= \cos x \cos 2x - 2\sin x \sin 2x$$

Exercises *differentiate by rule* 由公式求微分

對下列各函數進行微分

54. $x^2e^{2x^2+3}$ **:** $y = x^2e^{2x^2+3} = u \times v$

$$u = x^2,\ \frac{du}{dx} = 2x; \quad v = e^{2x^2+3},\ \frac{dv}{dx} = 4xe^{2x^2+3}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x^2 \times 4xe^{2x^2+3} + e^{2x^2+3} \times 2x$$

$$= 2x(1+2x^2)e^{2x^2+3}$$

Exercises *differentiate by rule* 由公式求微分

對下列各函數進行微分

55. $\dfrac{3x^2}{(2+x^2)^{1/2}}$ **:** $y = 3x^2(2+x^2)^{-1/2} = u \times v$

$$u = 3x^2,\ \frac{du}{dx} = 6x; \quad v = (2+x^2)^{-1/2},\ \frac{dv}{dx} = -x(2+x^2)^{-3/2}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = 3x^2 \times (-x)(2+x^2)^{-3/2} + 6x \times (2+x^2)^{-1/2}$$

$$= \frac{-3x^3}{(2+x^2)^{3/2}} + \frac{6x}{(2+x^2)^{1/2}} = \frac{3x(x^2+4)}{(2+x^2)^{3/2}}$$