## UNIT 3

# APPLICATION OF FINITE DIFFERENCE METHODS TO THE SOLUTION OF CONSOLIDATION PROBLEMS

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Note: During the 1940's and 1950's, consolidation topics concentrated on classical analyses of the Terzaghi type. Subsequently, finite difference methods were found to be much more useful. Finite difference analyses became increasingly sophisticated and hard to program, as we tried to find ways of reducing the computer time needed for the analyses. More recently, we have begun to return to the simpler methods, in the belief that we should reduce the programming effort and let high-speed computers do the work.

In addition, the amount of knowledge in geotechnical engineering has expanded steadily and we have tried to pack that knowledge into our courses. Clearly, we have to cut back somewhere. One place we will cut back, is in our discussion of finite difference methods. The notes will contain material that should be studied as usual, written in the Times New Roman font which has been used up to this point. Material provided for background, and which you can skip over, will be written in the Arial font, as is this sentence.

You will probably be best served if you read the Times New Roman text and understand it thoroughly, first. If time permits, you may want to look at the Arial material.

### **5.1 Introduction**

In analyzing consolidation problems, the geotechnical engineer begins by making a series of simplifying assumptions. Such assumptions may involve approximations of thicknesses of soil layers, assumptions that the soil is saturated, that soil properties are constant, strains are small, and so on. If a reasonable series of simplifying assumptions reduces the problem to one that has been solved theoretically, then a "solution" is easily obtained.

Realistic problems typically involve a series of soil layers, time dependent loading, variable coefficients of consolidation, and non-linear stress-strain curves, and may involve partial saturation, large strains, and a variety of other deviations from the simple theories we have discussed so far.

Considering the uncertainties obviously inherent in analysis of real foundation problems, the engineer may prefer an analytical technique that is less mathematically rigorous than those previously considered but one that can be applied to field problems using fewer simplifying assumptions.

"Approximate" solutions for wide variety of boundary value problems can be obtained using numerical methods of analysis. The numerical methods involve making a series of approximations to arrive at a final answer. Use of gross simplifications leads to rapid solution of the problem, but less accuracy. Gradual refinement of the method leads to increased accuracy at the expense of computational effort. In the extreme, it is usually possible to obtain solutions that are far more "accurate" than is the knowledge of the boundary conditions, initial conditions, or soil properties. The problem then becomes one of choosing the numerical procedure that yields a solution of acceptable accuracy with a minimum of computational effort.

In the case of one-dimensional consolidation problems, a numerical procedure of great practical value utilizes finite difference approximations. A variety of finite difference techniques may be used to solve a specific problem. We will emphasize the simple **explicit method** but provide information on use of the more complex **Crank-Nicholson method** as well.

General discussions of finite difference techniques have been presented by Salvadori and Baron (1952), Conte (1965), Hildebrand (1968), and many others.

A brief discussion of finite differences will be presented and then finite difference solutions for a variety of one-dimensional consolidation problems will be discussed

### **5.2 Finite Differences**

Finite differences may in general be divided into **forward differences**, **central differences**, and **backward differences**. In the following discussion, all three differences will be considered, but only to the extent necessary for specific application later in these notes. Because the finite differences may be applied to various dependent and independent variables, it is convenient to present the general discussion using y as the dependent variable and x as the independent variable. A curve showing y as a function of x is presented in Fig. 1. The curve can have any desired shape, the one chosen is arbitrary. We are concerned with specific points on the curve, termed **node points**, or just **nodes**. In particular, we are concerned with the points in the vicinity of an arbitrary point i. The coordinates of the curve at this point are  $x_i, y_i$ . Node points to the right have coordinates  $x_{i+1}, y_{i+1}; x_{i+2}, y_{i+2}; ..., x_{i+n}, y_{i+n}$ , and node points to the left have the corresponding coordinates  $x_{i-1}, y_{i-1}; x_{i-2}; ..., x_{i-n}, y_{i-n}$ . As a matter of convenience, the spacing of node points along the x axis is taken as uniform and equal to h.

### **5.2.1 Forward differences**

<u>Definition of first forward difference</u>. The symbol  $\Delta$  is used to indicate a forward difference. By definition, the first forward difference is:

$$\Delta \mathbf{y}_i = \mathbf{y}_{i+1} - \mathbf{y}_i \tag{5.1}$$

<u>Higher order forward differences</u>. Successive forward differences are obtained by multiplying through by the operator  $\Delta$ . Thus, the second forward difference becomes:

$$\Delta^{2} y_{i} = \Delta y_{i+1} - \Delta y_{i} = (y_{i+2} - y_{i+1}) - (y_{i+1} - y_{i})$$
  
=  $y_{i+2} - 2y_{i+1} + y_{i}$  (5.2)

This process may be continued indefinitely to obtain higher order forward differences.

<u>Taylor series solution</u>. Numerical values for the dependent variable y at successive node points may be obtained using a Taylor series. Thus, if  $y_i$  and all derivatives of  $y_i$  are known,  $y_{i+1}$  is given by the equation:

$$y_{i+1} = y_i + \frac{h}{1!} y_{i'} + \frac{h^2}{2!} y_{i''} + \dots + \frac{h^n}{n!} y_{i}^{n'} \dots$$
  
=  $y_i + \sum_{n=1}^{\infty} \frac{h^n}{n!} y_{i}^{n'}$  (5.3)

where y' indicates  $dy_i/dx$ , y" is  $d^2y_i/dx^2$ ,....

Equation 5.3 is now used to evaluate  $\Delta y_i$  in Eq. 5.1:

$$\Delta y_i = \frac{h}{1!} \quad y_i' + \frac{h^2}{2!} \quad y_i'' + \dots$$
(5.4)

As a first approximation, the Taylor series of Eq. 5.4 is truncated at the first term. The first foward difference is then written:

$$\Delta y_i = hy_i' + O(h^2) \tag{5.5}$$

where  $O(h^2)$  is a symbolic statement that the error involved in the truncation is of the order of  $h^2$ . The size of the error associated with the truncation of the series, is reduced as the power of h increases.

<u>Standard finite difference approximation for y'</u>. Based on Eq. 5.5, the first foward difference approximation for y' is:

$$y_i' = \frac{dy_i}{dx} = \frac{\Delta y_i}{h} = \frac{y_{i+1} - y_i}{h}$$
(5.6)

Equation 5.6 could have been "derived" by inspection (it applies to a straight line from point i to point i+1) but the Taylor series expansion is needed for more accurate approximations, for other types of difference equations, and because it provides an estimate of the error (here  $O(h^2)$ ).

Equation 7.6 is one of the approximations used in solving the differential equation for one dimensional consolidation; it is used for  $\partial \overline{u} / \partial t$ .

<u>More accurate approximations</u>. A more accurate estimate of y' can be obtained in the following way. Use the Taylor series to evaluate  $y_{i+2}$  and  $y_{i+1}$  in Eq. 5.2. The resulting equation for  $\Delta^2 y_i$  is then:

$$\Delta^{2} y_{i} = h^{2} y_{i}'' + h^{3} y_{i}''' + (\frac{7}{12}) h^{4} y_{i}''' - \dots$$

$$= h^{2} y_{i}'' + O(h^{3})$$
(5.7)

The terms involving  $h^2y''$  are then eliminated from Eqs. 5.7 and 5.4 to obtain:

$$2\Delta y_{i} - \Delta^{2} y_{i} = 2hy_{i}' - \frac{2}{3}h^{3} y_{i}''' - \frac{1}{2}h^{4} y_{i}''''$$
$$= 2hy' + 0 (h^{3})$$
(5.8)

Equation 5.8 yields an estimate of y' with an error that is of the order of  $h^3$ . Equations 5.1 and 5. 2 are substituted into Eq. 5.8, which is then solved for y' to obtain:

$$y_i' = \frac{dy_i}{dx} = \frac{-3y_i + 4y_{i+1} - y_{i+2}}{2h}$$
 (5.9)

Equation 5.9 gives a more accurate estimate of y' than does Eq. 5.6, but at the expense of more effort in the calculation.

In the following discussion of central and backward differences, only those forms actually used in solving consolidation problems will be presented. In every case, however, there will be other forms of the equations that yield more accurate solutions, but at the expense of more computational effort. In solving consolidation problems it is more convenient, usually, to attain satisfactory accuracy by reducing the value of h than by seeking more accurate forms of the finite difference equations.

#### **5.2.2 Central differences**

<u>First central difference approximation</u>. The first central difference is defined for the range in the variable x of h, as before, but this time centered over the point i. Thus, the first central difference,  $\delta y_{i}$ , is:

$$\delta y_i = y_{i+1/2} - y_{i-1/2} \tag{5.10}$$

However, the dependent variable, y, is defined only at node points. Thus, it is necessary to approximate the variable at the half node points as the average of the values at the adjacent node points. The "averaged first central difference" this then:

$$\mu \delta y_i = \frac{1}{2} (y_{i+1} + y_i) - \frac{1}{2} (y_i + y_{i-1}) = \frac{1}{2} (y_{i+1} - y_{i-1})$$
(5.11)

where  $\mu$  indicates an averaged value.

The Taylor series is again used to evaluate y at the two adjacent node points and the first averaged centeral difference becomes:

$$\mu \delta y_i = h y_i' + \left(\frac{1}{6}\right) h^3 y''' +$$
(5.12)

As a first approximation, the series is truncated after the first term, thus introducing an error of the order of  $h^3$ , and Eq. 5.12 is solved to obtain:

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$$
(5.13)

Equation 5.13 is the first averaged central difference approximation for y'. It has an accuracy  $O(h^3)$  and is thus more accurate than is the first forward difference, a conclusion that follows directly by inspection of Fig. 5.1.

Further, it may be seen by inspection of Fig. 5.1 that a difference approximation for y' is more accurate if it uses nodes arranged symmetrically around the i<sup>th</sup> node than if assymmetrical nodes are used.

<u>Second central differences</u>. The second central difference is found by multiplying Eq. 5.10 by the central difference operator,  $\delta$ , to obtain:

$$\delta^2 y_i = \delta y_{i+1/2} - \delta y_{i-1/2} = y_{i+1} - 2y_i + y_{i-1}$$
(5.14)

Again, the Taylor series is used to evaluate the dependent variable at adjacent node points, the series solution is truncated at the end of the first term, introducing an error of the order of  $h^4$ , and the resulting equation is set equal to Eq. 5.14 to obtain:

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$
(5.15)

Equation 5.15 will be used in consolidation equations as an approximation for  $\partial^2 u/\partial z^2$ .

<u>Backward Differences.</u> The first backward difference is defined as (Note: the proper symbol is an upside down delta but I don't have that symbol in this word processor so I'll use the § sign ad interim):

$$Sy_i = y_i - y_{i-1}$$
 (5.16)

The procedure used with forward differences is applied and the first backward difference approximation for y' is:

$$y' = \frac{y_i - y_{i-1}}{h}$$
(5.17)

### 5.3 Finite Difference Solution for a One Dimensional Consolidation Problem with Instantaneous Loading and Double Drainage Using the Explicit Method

#### **5.3.1 Solution for excess pore water pressures**

The finite difference equations will now be applied to solve the problem of a doubly drained clay layer, undergoing one dimensional consolidation. The assumptions made in deriving Terzaghi's theory are assumed valid. The differential equation to be solved is:

$$\frac{\partial \overline{u}}{\partial t} = c_v \frac{\partial^2 \overline{u}}{\partial z^2}$$
(5.18)

The rate of change of excess pore pressure with time is approximated using the first forward difference equation (Eq. 5.6) and the second partial derivative of the excess pore pressure with respect to depth is approximated using the second central difference equation (Eq. 5.15). Because the excess pore water pressure is a function of two variables, it is necessary to utilize two subscripts. The differential equation may then be written in either of the two following forms:

$$\frac{\overline{u}_{z,t+\Delta t} - \overline{u}_{z,t}}{\Delta t} = c_v \frac{\overline{u}_{z+\Delta z,t} - 2\overline{u}_{z,t} + \overline{u}_{z-\Delta z,t}}{(\Delta z)^2}$$
(5.19a)

$$\frac{\overline{u}_{i,j+1} - \overline{u}_{i,j}}{\Delta t} = c_v \frac{\overline{u}_{i+1,j} - 2\overline{u}_{i,j} + \overline{u}_{i-1,j}}{(\Delta z)^2}$$
(5.19b)

Equation 5.19a is generally used for purpose of discussion and hand solution, whereas Eq. 5.19b is used for solutions obtained using the digital computer. Equation 5.19a may be solved for  $u_{z,t+\Delta t}$  to obtain:

$$u_{z,t+\Delta t} = u_{z,t} + \frac{c_{v}\Delta t}{(\Delta z)^{2}} \left[ u_{z-\Delta z,t} - 2u_{z,t} + u_{z+\Delta z,t} \right]$$
(5.20)

The term  $c_v \Delta t/(\Delta z)^2$  is usually replaced by a coefficient  $\alpha$  to simplify both writing and discussion.

According to Eq. 5.20, if the excess pore water pressure is known at time t at point z, and at the points just above and just below z, then the pore pressure can be calculated at the point z at a time  $\Delta t$  later than t. If such a calculation can be made for any depth z and for any time t, then it can be made throughout the clay layer for all times ranging from zero to infinity and complete pore pressure isochrones can be calculated. From the pore pressures, the degrees of consolidation and settlements can be calculated.

#### **5.3.2** Comparison with Terzaghi's solution

For solutions of real problems it is convenient to use real time, t, and real node spacing,  $\Delta z$ , in Eq. 5.20. However, in our initial studies we are interested in comparing our numerical solutions with Fourier series solutions so we would like to convert from real depth and real time to dimensionless values. From the definition of time factor, T:

$$\Delta t = \frac{H^2}{c_v} \quad \Delta T \tag{5.21}$$

It is convenient to define a dimensionless depth,  $\zeta$ , as:

$$\zeta = \frac{z}{H} \tag{5.22}$$

Insertion of Eqs. 5.21 and 5.22 into Eq. 5.20 leads to:

$$\overline{u}_{\zeta,T+\Delta T} = \overline{u}_{\zeta,T} + \frac{\Delta T}{\left(\Delta\zeta\right)^2} \left[ \overline{u}_{\zeta-\Delta\zeta,T} - 2\overline{u}_{\zeta,T} + \overline{u}_{\zeta+\Delta\zeta,T} \right]$$
(5.23)

where  $\Delta T/(\Delta \zeta)^2$  is again  $\alpha$ . Equation 5.23 will now be applied, by hand, to solve a consolidation problem.

The initial excess pore water pressure is assigned a value of 100. It is found that successive application of Eq. 5.23 soon leads to ridiculous values of pore water pressures (is divergent) if  $\alpha$  exceeds 0.5 (this may be proved theoretically too). As a matter of convenience in the hand solution, let  $\alpha = 0.25$ . The accuracy of the solution is improved by using a small spacing of node points in the z direction. However, to simplify the hand solution choose  $\Delta \zeta = 1/5$ . Thus, the increment of time factor is:

$$\Delta T = \alpha (\Delta \zeta)^2 = (0.25)(0.2)^2 = 0.01$$

The method of making the calculations is shown in Table 5.1 and the arrangement of node points in Fig. 5.2. At time zero the excess pore water pressures are all 100 except at the drainage boundaries where they are simultaneously 100 and 0. An average initial excess pore pressure of 50 is used at the drainage boundaries but for all subsequent times this pore

T=	0	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10
Node 1											
	50	0	0	0	0	0	0	0	0	0	0
2	100	88	69	59	52	47	44	41	39	37	35
3	100	100	97	91	85	80	76	72	68	66	64
4	100	100	100	99	97	95	92	90	87	85	83
5	100	100	100	100	100	99	98	97	96	94	92
6	100	100	100	100	100	100	100	98	97	97	95
7	100	100	100	100	100	99	98	97	96	94	92

Table 5.1 - Hand Calculation of Pore Pressures using Finite Differences

pressure is zero. As an example of the application of the equation, the excess pore water pressure at the first node point and at a time factor of 0.01, u(0.20, 0.01) is:

$$\overline{u}(0.20, 0.01) = 100 + (\frac{1}{4})[50 - (2)(100) + 100] = 88$$

Values of the pore pressure at a time factor of 0.01 are calculated for successively lower node points until the lower boundary is reached. We note, however, that the double drained clay layer is symmetrical about the mid-depth, the sixth node point. The seventh node point is the same as the fifth and really need not be shown if one just remembers that when the node point at mid-depth is used in the calculation, the 7th and 5th node points are identical. It is

also apparent that the presence of an impervious boundary would simply be taken into account by placing an imaginary node point on the outside (away from the compressible layer) that would have a pore pressure the same as that of the first node point within the compressible layer and proceeding with the calculations as indicated above.

A simple check on the accuracy of the foregoing finite-difference analysis is to compare the isochrones calculated by using finite differences with those calculated with the Fourier series solution. Such a comparison, for several values of time factor, is shown in Fig. 5.3. Apparently even this relatively crude finite difference analysis gives good estimates of the isochrones.

### 5.3.3 Average degree of consolidation

<u>Definition</u>. The average degree of consolidation, U, is defined as:

$$U = \frac{\int u_i \, dz - \int u \, dz}{\int u_i \, dz}$$
(5.24)

For the rectangular stress surface:

$$\int u_i dz = u_i(2H)$$

where the layer thickness is 2H.

The area under an isochrone,  $\int u dz$ , must be obtained by graphical integration.

<u>Trapezoidal integration</u>. The simplest method of integration is to use trapezoidal integration, i.e, to assume the pore pressures vary linearly from node to node. The area under the isochrone between the i<sup>th</sup> and (i+1)<sup>th</sup> nodes, spaced a distance  $\Delta \zeta$  apart, is then:

$$A = (\Delta \zeta)(1/2)(u_i + u_{i+1})$$
(5.25)

The area under the entire isochrone is then:

$$\int u dz \approx \frac{1}{2} \Delta \zeta (u_1 + 2u_2 + 2u_3 + \dots 2u_{N-1} + u_N)$$
 (5.26)

where there are N nodes. If trapezoidal integration is used, then the area under the isochrone at T = 0.10 (Table 1) is 64.3. The degree of consolidation is then (100-64.3)/100 = 35.7%. The Fourier series solution gives 35.7% also.

<u>Simpson's rules</u>. A somewhat more accurate integration scheme is to use Simpson's one-third rule with:

$$\int u dz = \frac{1}{3} \Delta \zeta \left( u_1 + 4u_2 + 2u_3 + 4u_4 + \dots 2u_{N-2} + 4u_{N-1} + u_N \right)$$
(5.27)

When Simpson's one-third rule is used throughout, N must be an odd number that is greater than two. Alternatively, the analysis can use the one-third rule down to the last space and then use the trapezoidal rule. As a third option, Simpson's one-third rule can be used until there are three spaces left and then Simpson's three-eighths rule can be applied:

$$\int u dz = \frac{3}{8} \Delta \zeta \left( u_1 + 3u_2 + 3u_3 + 2u_4 + 3u_5 \dots 2u_{N-3} + 3u_{N-2} + 3u_{N-1} + u_N \right)$$
(5.28)

The accuracy of finite difference solutions depends on the number of node points, N, and on  $\Delta t$ . The accuracy of the finite difference method of analysis is indicated in Figs. 5.4 and 5.5. For these curves, Simpson's one third rule was used for integration.

### 5.4 Finite Difference Solution for a One Dimensional Consolidation Problem with Instantaneous Loading, Single or Double Drainage, and Using the Crank-Nicholson Method

The explicit method has the advantage of being simple to understand but the restriction on values of  $\alpha$  is inconvenient. An alternative method, the Crank-Nicholson method, is somewhat more difficult to set up but is more powerful for solving complicated problems.

In the Crank-Nicholson method (hereafter abbreviated to the C-N method), the first forward difference is again used for  $\partial u/\partial t$ . We note that this forward difference represents the approximate average value of  $\partial u/\partial t$  for the time period T(J) to T(J+1). In the explicit method,  $\partial^2 u/\partial z^2$ , was approximated at the time T(J), not as the average between times T(J) and T(J+1), thus resulting in the two sides of the differential equation being approximated at slightly different times, and leading to instability. The C-N method uses the second central difference approximation for  $\partial^2 u/\partial z^2$ , but it uses the average value of this approximation at T(J) and T(J+1). Thus:

$$\left(\frac{\partial^2 u}{\partial z^2}\right)_{j} = \frac{1}{(\Delta z)^2} (u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$
(5.29a)

$$\left(\frac{\partial^2 \mathbf{u}}{\partial z^2}\right)_{j+1} = \frac{1}{(\Delta z)^2} (\mathbf{u}_{i-1,j+1} - 2\mathbf{u}_{i,j+1} + \mathbf{u}_{i+1,j+1})$$
(5.29b)

and the average value of  $(\partial^2 u / \partial z^2)$  is:

$$\frac{\partial^2 u}{\partial z^2} = \frac{1}{2(\Delta z)^2} (u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1})$$
(5.30)

Equation 5.30 is combined with the first forward difference approximation for  $\partial u/\partial t$  to convert Terzaghi's differential equation for one dimensional consolidation (Eq. 5.18) into the form:

$$\frac{\overline{u}_{i,j+1} - \overline{u}_{i,j}}{\Delta t} = \frac{c_{\rm v}}{2(\Delta z)^2} (u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1})$$
(5.31)

Again, the term  $c_v \Delta t/(\Delta z)^2$  is replaced with  $\alpha$ . The resulting equation possesses, in general, three unknowns, viz.,  $u_{i-1,j+1}$ ,  $u_{i,j+1}$ , and  $u_{i+1,j+1}$ . It is convenient to rewrite Eq. 5.31 as follows:

$$-u_{i-1,j+1} + \frac{2(1+\alpha)}{\alpha} u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + \frac{2(1-\alpha)}{\alpha} u_{i,j} + u_{i+1,j} = C_i$$
(5.32)

In the case of a doubly drained compressible layer with N depth nodes, the excess pore water pressures at the boundary nodes are zero, and the excess pore water pressure at the N-2 interior nodes are to be calculated. Since Eq. 5.32 can be written separately for each of the interior nodes, it is possible to write N-2 equations containing these N-2 unknown pore pressures, and thus to solve for the pore pressures. One important advantage of using this method is that  $\alpha$  is not limited to the range of 0 to 1/2 but may be considerably larger.

As an example of a hand solution, the problem shown in Fig. 5.2 will be reanalyzed, usng  $\alpha = 1/2$  and eleven nodes in the range from 0 to 2H. Thus,  $\Delta z = 0.20$  and the iteration time factor,  $\Delta T$ , is (1/2)(2/10)2 = 0.020. From Eq. 5.32:

$$-u_{i-1,j+1} + 6u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + 2u_{i,j} + u_{i+1,j}$$
(5.33)

Because of symmetry about the mid-depth of the doubly drained compressible layer,  $u_{5,j} = u_{7,j}$ ,  $u_{4,j} = u_{8,j}$ , . . . , for all values of j. At node one  $u_{1,j+1} = u_{1,j} = 0$ . The equations at successive nodes become:

i = 2	6u <sub>2</sub> - u <sub>3</sub>	= 350	
i = 3	-u <sub>2</sub> + 6u <sub>3</sub> - u <sub>4</sub>	= 400	
i = 4	- u <sub>3</sub> + 6u <sub>4</sub> - u <sub>5</sub>	= 400	(5.34)
i = 5	- u <sub>4</sub> + 6u <sub>5</sub> - u <sub>6</sub>	= 400	
i = 6	- u <sub>5</sub> + 6u <sub>6</sub> - u <sub>7</sub>	= 400	

Since the isochrones are symmetrical about the mid-depth,  $u_7 = u_5$  and the last equation (i = 6) becomes:

$$i = 6$$
  $-2u_5 + 6u_6 = 400$ 

and calculations stop at the sixth node because of symmetry. Equation 5.34 may be rewritten in matrix form as:

$$\begin{bmatrix} 6-1 & 0 & 0 & 0 \\ -1 & 6-1 & 0 & 0 \\ 0 & -1 & 6-1 & 0 \\ 0 & 0 & -1 & 6-1 \\ 0 & 0 & 0-2 & 6 \end{bmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} = \begin{cases} 350 \\ 400 \\ 400 \\ 400 \\ 400 \end{cases}$$

or

$$\mathbf{A} \mathbf{u}_{\mathbf{i}} = \mathbf{C} \tag{5.35}$$

where A is the matrix of the coefficients,  $u_i$  is the column matrix of unknown pore pressures, and C is the column matrix of constants. The matrix A is termed a tridiagonal matrix because there are only three non-zero terms in each row and they are symmetrical about the main diagonal, except for the first and last rows.

### 5.4.1 Gauss-Seidel iteration

A method of solution of Eq. 5.35, that works very well for one dimensional consolidation problems, is the Gauss-Seidel iteration method. The procedure is as follows: First solve each equation for the main diagonal unknown:

$$u_{2} = 58.33 + (1/6)u_{3}$$
  

$$u_{3} = 66.67 + (1/6)u_{2} + (1/6)u_{4}$$
  

$$u_{4} = 66.67 + (1/6)u_{3} + (1/6)u_{5}$$
  

$$u_{5} = 66.67 + (1/6)u_{4} + (1/6)u_{6}$$
  

$$u_{6} = 66.67 + (1/3)u_{5}$$

Then, any set of initial values are assumed for the unknown pore pressures. The first equation is then solved for  $u_2$  in terms of the assumed value of  $u_3$ . Then the second equation is solved for  $u_3$  using the just calculated value for  $u_2$  and the assumed value of  $u_4$ . The remainder of the equations are solved, successively, in the same manner. Then the first equation is solved again, using the last calculated value of  $u_3$ , and the process is repeated, always using the last estimated value for each of the pore pressures.

Iterations continue until no unknown pore pressure changes by more than some negligibly small amount during one complete set of iterations. The method converges rather rapidly, especially if the main diagonal coefficient is a large number, as for small values of  $\alpha$ . To start the analysis, the pore water pressures at time zero are set equal to the values from the stress surface, thus satisfying the "initial conditions". At time  $\Delta t$ , the first approximations for excess pore water pressures are the values just set at time zero, except at drainage boundaries where pore pressures are reset to zero. For subsequent times, the first approximations of the excess pore water pressures are their most recently calculated values.

A set of hand calculations for Eq. 5.34 are shown in Table 5.2. Satisfactory convergence was achieved in about the third iteration.

u <sub>2</sub> =	u3 =	u4 =	u5 =	u <sub>6</sub> =	Iteration
58.33+	66.67+	66.67+	66.67+	66.67+	
u3/6	u <sub>2</sub> /6+	u <sub>3</sub> /6+	u <sub>4</sub> /6 +	u <sub>5</sub> /3	
	и4/б	u <sub>5</sub> /б	u <sub>6</sub> /б		
100.00	100.00	100.00	100.00	100.00	Initial
75.00	95.83	99.31	99.88	99.96	First
74.31	95.60	99.25	99.87	99.96	Second
74.27	95.59	99.24	99.87	99.96	Third
74.26	95.58	99.24	99.87	99.96	Fourth

Table 5.2Hand Solution of Crank-Nicholson Equations using Gauss-Seidel Iteration

A T-U curve calculated using the Gauss-Seidel method and 42 nodes is compared with the Fourier series T-U curve in Fig. 5.6. The Gauss-Seidel iterations were stopped when no pore pressure changed by more than 0.000001% of the initial pore pressure or when 50 iterations had occurred, which ever came first. Values of \_\_ranged from 0.80 to 160.

### 5.4.2 Gauss elimination.

The other commonly used method of solving Eq. 5.35 is Gauss elimination. The method is used to change the matrix of coefficients (matrix A of Eq. 5.35) into the upper triangular form. For example, for one particular problem, Eq. 5.34 might be converted into the form:

 $D_{2}u_{2} - u_{3} = C_{2}$   $D_{3}u_{3} - u_{4} = C_{3}$   $D_{4}u_{4} - u_{5} = C_{4}$   $D_{1}u_{1} - u_{1+1} = C_{1}$   $D_{N-1}u_{N-1} - u_{N} = C_{N-1}$   $D_{N}u_{N} = C_{N}$   $D_{N}u_{N} = C_{N}$ 

Once in this form, the equations are solved by back substitution:

 $u_{N} = C_{N}/D_{N}$   $u_{N-1} = (C_{N-1} + u_{N})/D_{N-1}$   $u_{1} = (C_{1} + u_{1+1})/D_{1}$   $u_{2} = (C_{2} + u_{3})/D_{2}$   $u_{1} = 0$ (5.37)

Let us use Eq. 5.34 as an example. The original equations are:

Eq.1	u <sub>1</sub> =	0
2	6u <sub>2</sub> -u <sub>3</sub> =	350
3	$-u_2+6u_3-u_4=$	400
4	$-u_3 + 6u_4 - u_5 =$	400
5	$-u_4 + 6u_5 - u_6 =$	400
б	-u <sub>5</sub> +6u <sub>6</sub> -u <sub>7</sub> =	400
7	$-u_{6}+6u_{7}-u_{8}=$	400
8	-u <sub>7</sub> +6u <sub>8</sub> -u <sub>9</sub> =	400
9	-u <sub>8</sub> +6u <sub>9</sub> -u <sub>10</sub> =	400
10	-u <sub>9</sub> +6u <sub>10</sub> =	350
11	u <sub>11</sub> =	0

- 1. Divide Eq. 2 by 6 to obtain  $u_2$ -0.1667 $u_3$  = 58.333
- 2. Add Eqs. 3 and (new) 2 to obtain 5.833u<sub>3</sub>-u<sub>4</sub>=458.333
- 3. Divide this equation by 5.833 to obtain  $u_3-0.1714u_4 = 78.576$
- 4. Substitute this equation for the previous Eq. 3
- 5. Add this (new) Eq. 3 to Eq. 4 to obtain: 5.8286u<sub>4</sub>-u<sub>5</sub>=478.576
- 6. Divide by 5.8286 to obtain: u<sub>4</sub>-0.1716u<sub>5</sub>=82.108
- 7. Substitute this equation for the previous Eq. 4
- 8. Add this (new) Eq.4 to Eq. 5 etc.

In their reduced form, the equations are (remember  $u_1=u_{11}=0.0$ ):

- $u_2-0.1667u_3=58.33$   $u_3-0.1714u_4=78.576$   $u_4-0.1716u_5=82.108$   $u_5-0.1716u_6=82.717$   $u_6-0.1716u_7=82.822$   $u_7-0.1716u_8=82.8523$   $u_8-0.1716u_9=82.857$   $u_9-0.1716u_{10}=82.8455$   $5.8284u_{10}=432.8455$ Backsubstitution then yields:  $u_{11}= 0.00$   $u_{10}= 74.26$ 
  - $u_{10} = 74.26$   $u_{9} = 95.20$   $u_{8} = 99.19$   $u_{7} = 99.87$   $u_{6} = 99.96$   $u_{5} = 99.87$   $u_{4} = 99.24$   $u_{3} = 95.58$   $u_{2} = 74.26$  $u_{1} = 0.00$

These pore water pressures are essentially the same as those calculated using the Gauss-Seidel method (Table 5.2).

### 5.4.3 Time steps for the Crank-Nicholson method

There is no need to use the same time step  $\Delta T$  for all times when the Crank-Nicholson method is used. Instead, efficiency is improved by specifying the times for which output of data is desired and then iterating directly from one output time to the next. As the iteration

time,  $\Delta T$ , increases the solution becomes progressively less accurate but it never becomes unstable as in the explicit method. To ensure a reasonable accuracy the programmer may set some type of limit on the value of  $\alpha$  such that if the iteration time  $\Delta T$  leads to too large a value for  $\alpha$  then the computer will pick intermediate times for one or more calculations and will then go to the output time. Limits can be set on  $\alpha$  that depend on certain other factors. For example, it is found that  $\alpha$  must be limited to small values right after a load is applied but once the isochrones become reasonably smooth shaped the values of  $\alpha$  may be increased substantially (I generally limit  $\alpha$  to 2 but have had success on occasion with values as high as 100).

### 5.5 Time Dependant Loading

One dimensional consolidation problems of practical interest always involve time dependent loading. Thus, the first attempt to generalize the simple Terzaghi solution will involve a time dependent variation in applied stresses. The applied stress is assumed to be known at discrete points, and to vary linearly in between these points. The differential equation of one dimensional consolidation with a variable applied stress, q, is:

$$\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2} + \frac{dq}{dt}$$
(5.38)

### 5.5.1 Solution using the Explicit Method

The explicit method may be applied to solve Eq. 5.38 by using the first forward difference to represent the change in the surface load. If the pore pressure generated by the applied load is independent of depth the equation to be solved assumes the form:

$$u_{i,j+1} = u_{i,j} + \alpha (u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + (q_{j+1} - q_j)$$
(5.39)

Thus, the new pore pressures are calculated from the previous values just as for the case of no variation of surface load with time, and then the change in load during the time  $\Delta T$  is added in. The full value  $q_{j+1} - q_j$  is added at each interior node and  $(1/2)(q_{j+1} - q_j)$  is added for nodes at freely draining boundaries.

The variation of applied load with time is conveniently input to the computer by specifying the coordinates of points on the load versus time curve, as shown in Fig. 5.7. The load at any time is then found by linear interpolation between points on the load-time curve

To demonstrate the accuracy of the method, a solution was obtained for the case of a ten-foot thick clay layer, with single drainage and a vertical coefficient of consolidation of 0.2 ft<sup>2</sup>/day, subjected to a fill load that was of the single ramp loading type with a construction time of 30 days. The time-settlement curve obtained using finite differences with  $\alpha = 1/6$  and 21 nodes is compared with the exact solution in Fig. 5.8.

### 5.5.2 Crank-Nicholson method

The Crank-Nicholson equation is again Eq. 5.32. However, at the end of each iteration time the change in load for the next time step is added as an instantaneous load.

For purposes of both discussion and programming it is convenient to make the following definitions:

T(J) = the previous calculation time

TC = the calculation time, previously indicated by the subscript j+1

TO(JO) = the next time for which we desire data on pore pressures or settlement

TL(JL) = the time coordinate of the next point on the load-time curve

The program might then use the following FORTRAN statement:

which says, in words, to jump ahead to the next point at which loading conditions are defined, or there is a required output time. The resulting value of  $\alpha$  is calculated for each soil layer and, if it exceeds a user-input limit, say ALPLIM, then the time step is reduced accordingly.

### **5.6 Layered Systems with Constant Soil Properties**

The soil deposit is now considered to consist of a sequence of strata with horizontal boundaries, subject to one-dimensional consolidation with vertical flow only. It will be assumed that none of the properties, nor the layer thicknesses, change significantly during consolidation. Solutions can easily be developed for cases of partially draining boundaries but it is probably better to treat relatively incompressible layers as just additional layers in the system rather than as partially draining boundaries.

### 5.6.1 Explicit method

<u>Restrictions within layers</u>. It is necessary to make the iteration time,  $\Delta t$ , the same for all layers in the system. Thus, from the definition of  $\alpha$ :

$$\Delta t = \alpha_1 (\Delta z_1)^2 / c_{v1} = \alpha_2 (\Delta z_2)^2 / c_{v2} = \dots = \alpha_N (\Delta z_N)^2 / c_{vN}$$
(5.40)

The coefficient of consolidation of each layer is known in advance, as is the layer thickness. The engineer must then choose values appropriate values of  $\Delta z$ , and then use values of  $\alpha$  that differ from layer to layer but which satisfy Eq. 5.40.

The time step is controlled by the layer with the highest coefficient of consolidation and the smallest thickness. For the unwary analyst, disaster strikes if there is a thin seam of sand because the time step for such a layer is essentially zero. However, it is obvious that there will be no gradient across a thin seam of sand in between two thick clay layers so the analysis can proceed by requiring that the sand layer have only two nodes, one at the top and one at the bottom, and that they have the same pore water pressure. Of course, the program cannot ignore the effect of the unit weight of the sand on the stresses.

The analyst may have to use care with layers that are still clay but have a much higher  $c_v$  than other layers because they can result in horrendous execution times. In my programs, I have the program calculate the initial time step for each layer using the upper limit on  $\alpha$ , and then ask the user if the number of nodes in any layers should be reset.

Let us assume that the engineer has chosen values of  $\alpha$ ,  $\Delta z$ ,  $c_v$ , and H (layer thickness), that are consistent with Eq. 5.40. Equation 5.20 may now be applied for all nodes that are within individual layers, i.e., not at interfaces between layers. At the interface it becomes necessary to develop a new equation.

At the interface between two contiguous compressible layers, the flow from one layer must equal the flow into the other layer. If the subscript v is used to indicate the upper layer and w the lower layer, then from Darcy's law:

$$q = \frac{k_{v}}{\gamma_{w}} \left( \frac{\partial u}{\partial z} \right) {}_{v} A = \frac{k_{w}}{\gamma_{w}} \left( \frac{\partial u}{\partial z} \right) {}_{w} A$$

Thus:

$$k_{v}\left(\frac{\partial u}{\partial z}\right)_{v} = k_{w}\left(\frac{\partial u}{\partial z}\right)_{w}$$
(5.41)

The simplest way to insert this requirement into the analysis is to use a backward difference for  $(\partial u/\partial z)_V$  and a forward difference for  $(\partial u/\partial z)_W$ . If the i<sup>th</sup> node is at an interface, then:

$$k_{v} \frac{u_{i} - u_{i-1}}{\Delta z_{v}} = k_{w} \frac{u_{i+1} - u_{i}}{\Delta z_{w}}$$
(5.42)

Since Eq. 5.42 applies at all times, it is not necessary to use a subscript j to indicate time. As a matter of convenience, define:

$$\beta_{\rm v} = k_{\rm v} / \Delta z_{\rm v} \tag{5.43a}$$

$$\beta_{\rm W} = k_{\rm W} / \Delta z_{\rm W} \tag{5.43b}$$

Equation 5.42 is now solved for the pore pressure at the interface, u<sub>i</sub>, to obtain:

$$u_{i} = \frac{\beta_{w}u_{i+1} + \beta_{v}u_{i-1}}{\beta_{v} + \beta_{w}}$$
(5.44)

We now assume that all the pore pressures are known at some time t. To calculate the pore pressures at time  $t + \Delta t$ , we first apply the appropriate equations, e.g., Eq. 5.20, for all nodes not at a boundary or interface. Boundary pore pressures are here assumed either to be zero (freely draining boundary) or calculated with an imaginary exterior node (impervious boundary). Finally, Eq. 5.44 is applied for nodes at interfaces. All values of u are then known. The process is repeated for successive times.

The foregoing method of evaluating pore pressures at interface nodes has the advantage of simplicity but, as noted earlier, use of first forward or backward differences is not as accurate as using a central difference. The first averaged central difference approximation for Eq. 5.41 is:

$$k_{v} \frac{\overline{u}_{i+1} - \overline{u}_{i-1}}{2\Delta z_{v}} = k_{w} \frac{\overline{u}_{i+1} - \overline{u}_{i-1}}{2\Delta z_{w}}$$
(5.45)

Although this approximation is more accurate, it suffers from the defect that the pore pressure  $(u_{i+1})_v$  exists only if the layer v were to continue into the region actually occupied by layer w. Similarly, the pore pressure  $(u_{i-1})_w$  exists only if the layer w were to be extended backwards into the region occupied by layer v. Thus, we have two imaginary pore pressures. To make it clear which pore pressures are imaginary we will denote  $(u_{i+1})_v$  by  $v_{i+1}$  and  $(u_{i-1})_w$  by  $w_{i-1}$ . It is also convenient to substitute:

$$\gamma = \frac{k_{\rm W} \Delta z_{\rm V}}{k_{\rm V} \Delta z_{\rm W}} \tag{5.46}$$

Equation 5.45 then becomes:

$$v_{i+1} - u_{i-1} = \gamma(u_{i+1} - w_{i-1})$$
(5.47)

Equation 5.47 contains two imaginary pore pressures which can be eliminated if we can develop two additional equations containing these imaginary pore pressures. We write the regular finite difference equations for the v and w layers as if both were continuous, with the ith node still at the interface.

$$u_{i,j+1} - u_{i,j} = \alpha_v (u_{i-1,j} - 2u_{i,j} + v_{i+1,j})$$
(5.48a)

$$u_{i,j+1} - u_{i,j} = \alpha_{w}(w_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$
(5.48b)

Equations 5.48a and 5.48b contain the two imaginary pore pressures but they must specifically be known at time j. Thus, we rewrite Eq. 5.47 to apply at time j (remember that Eq. 5.47 is valid at any time):

$$v_{i+1,j} - u_{i-1,j} = \gamma(u_{i+1,j} - w_{i-1,j})$$
(5.49)

We will assume that the engineer has chosen node spacings to make  $\alpha_v = \alpha_w$ . Equations 5.48a and 5.48b are solved for the imaginary pore pressures which are then inserted into Eq. 5.49 to obtain:

$$u_{i,j+1} = u_{i,j} + \frac{2\alpha}{1+\gamma} \left[ u_{i-1,j} - (1+\gamma)u_{i,j} + \gamma u_{i+1,j} \right]$$
(5.50)

All pore pressures and coefficients on the right side of Eq. 5.50 are known. Thus, the solution is obtained by evaluating all the pore pressures for nodes within compressible layers at each time j+1, and then immediately evaluating the pore pressures at interface nodes using Eq. 5.50. Thus, all pore pressures are known at time j+1 and we may proceed to the next set of iterations. If values of  $\alpha_v$  and  $\alpha_w$  are quite unequal, then it would be better to use Eq. 5.44.

In applying the foregoing equations to consolidation of layered systems, problems develop if layers have greatly different coefficients of consolidation and thicknesses. Let us re-examine the problem of selecting the number of nodes in individual layers. Let  $H_L$  indicate the thickness of the L<sup>th</sup> layer. Then:

$$(N_L-1)(\Delta z_L) = H_L \tag{5.51}$$

where  $N_L$  is the number of nodes in the L<sup>th</sup> layer, counting the nodes at the boundaries, and  $\Delta z_L$  is the node spacing. The iteration time is then:

$$\Delta t = \Delta t = \frac{H_L^2 \alpha_L}{c_{\nu L} (N_L - 1)^2}$$
(5.52)

It the iteration time is held constant for all layers then:

$$\frac{\alpha_{1}H_{1}^{2}}{c_{v1}(N_{1}-1)^{2}} = \frac{\alpha_{2}L_{2}^{2}}{c_{v2}(N_{2}-1)^{2}} = \dots = \frac{\alpha_{L}H_{L}^{2}}{c_{vL}(N_{L}-1)^{2}} = \dots = \frac{\alpha_{N}L_{N}^{2}}{c_{vN}(N_{N}-1)^{2}}$$
(5.53)

Equation 5.53 will be applied to any two layers which will be denoted as the L<sup>th</sup> and S layers. From Eq. 5.53:

$$N_{L} = 1 + \frac{H_{L}}{H_{S}}(N_{S}-1)\sqrt{\frac{\alpha_{L}c_{vS}}{\alpha_{S}c_{vL}}}$$
(5.54)

Suppose now that the layer S is a sand layer one inch thick and with a coefficient of consolidation  $10^8$  larger than that of an adjacent clay layer, which is say ten feet thick. If we put a minimum of three nodes in the sand layer, and use the same values of throughout, then from Eq. 5.54 the clay layer will have 2,400,001 nodes! The engineer can avoid such problems by simply ignoring the presence of sand layers, i.e., by analytically removing the sand layers from the soil profile and assuming that pore pressures within individual sand layers will be uniform. It is clear that for an included sand layer of small thickness, the ratio N<sub>L</sub>/N<sub>S</sub> becomes very large and an impossibly large number of nodes would be required for, say, a clay layer. The program must either be written to check for such a contingency, and simply ignore the sand layer, or the engineer must predetermine the number of nodes in each layer and ensure that the number is reasonable.

#### 5.6.2 Crank-Nicholson method

The notation used for the derivation of the explicit solution is retained. We again consider the problem when the  $i^{th}$  node is in an interface and the  $v^{th}$  layer is above the wth layer is below. The two Crank-Nicholson equations are:

$$-u_{i-1,j+1}+C_{Lv}u_{i,j+1}-v_{i+1,j+1}=u_{i-1,j}+C_{Rv}u_{i,j}+v_{i+1,j}$$
(5.55a)

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$$-w_{i-1,j+1}+C_{Lw}u_{i,j+1}-u_{i+1,j+1}=w_{i-1,j}+C_{Rw}u_{i,j}+u_{i+1,j}$$
(5.55b)

where:

$$C_{\rm L} = 2(1+\alpha)/\alpha \tag{5.55c}$$

and:

$$C_{\rm R} = 2(1-\alpha)/\alpha \tag{5.55d}$$

and  $v_{i+1,j}$ ,  $v_{i+1,j+1}$ ,  $w_{i-1,j}$ , and  $w_{i-1,j+1}$  are imaginary and must be eliminated. The continuity equations at the boundary at the times corresponding to j and j+1 are:

$$k_{v} \frac{\overline{v}_{i+1,j} - \overline{u}_{i-1,j}}{2\Delta z_{v}} = k_{w} \frac{\overline{u}_{i+1,j} - \overline{w}_{i-1,j}}{2\Delta z_{w}}$$
(5.55e)

$$k_{v} \frac{\overline{v}_{i+1,j+1} - \overline{u}_{i-1,j+1}}{2\Delta z_{v}} = k_{w} \frac{\overline{u}_{i+1,j+1} - \overline{w}_{i-1,j+1}}{2\Delta z_{w}}$$
(5.55f)

Equations 5.55a-f are solved simultaneously to eliminate the imaginary pore pressures. The resulting equation is simplified by introducing the variables:

$$\gamma = \frac{k_{\rm W}}{k_{\rm V}} \frac{\Delta z_{\rm V}}{\Delta z_{\rm W}}$$
(5.56a)

$$\phi_{\rm L} = \frac{1}{2} (C_{\rm Lv} + \gamma C_{\rm Lw}) \tag{5.56b}$$

$$\phi_{\rm R} = \frac{1}{2} (CRv + \gamma C_{\rm Rw}) \tag{5.56c}$$

The resulting equation for the boundary node is:

$$-u_{i-1j+1} + \phi_L u_{i,j+1} - \gamma u_{i+1,j+1} = u_{i-1,j} + \phi_R u_{i,j} + \gamma u_{i+1,j} = C_i$$
(5.57)

In the Crank-Nicholson solution, the equations discussed for a homogeneous layer are used for interior nodes and Eq. 5.57 is applied at each boundary node.

The Crank-Nicholson method is somewhat easier to apply to a layered system than is the explicit method, because of the wider range in values for  $\alpha$ .

### **5.7 Variable Coefficient of Consolidation**

The discussion of techniques for accounting for variable coefficients of consolidation, nonlinear stress-strain curves, and large strains, need not involve consideration of the particular finite difference scheme to be used, nor of whether or not the system is stratified and the loading time dependent, and none of these factors will be included in this discussion.

The finite difference method can be used only if the coefficient of consolidation is constant for the iteration period,  $\Delta t$ , but there is no requirement that the coefficient be the same from one iteration to the next. Thus, a simple method for accounting for variations in the coefficient of consolidation is to input into the computer program the plot of  $c_v$  versus  $\sigma'$  and to use the effective stresses at the end of any iteration time to read off the appropriate value of  $c_v$  for the next iteration. We find it convenient to input selected points on the plot of  $c_v$ [denoted as CVP(JCV)] in the program) versus  $\sigma'$  [denoted as PCV(JCV) in the program], as shown in Fig. 5.9a, and have the program determine the appropriate values of  $c_v$  by linear interpolation.

A problem develops immediately because the effective stress varies with depth even in a "homogeneous" layer, and thus  $c_v$  becomes a function of depth. We would find ourselves with each node representing a separate compressible layer. To avoid this problem, the soil is divided into a suitable number of layers and the properties of each layer are assumed to be uniform. An average effective stress for each layer is then used to determine the appropriate  $c_v$  for the next time period for all nodes within the layer.

It may also be noted that the coefficients of consolidation might change by a substantial amount during any time period, especially when the Crank-Nicholson method is used with a large value of  $\alpha$ . Our approach to this problem is as follows: First, the programs are written to contain a check on the fractional change (we call it CHG) in  $c_v$ . If the value of CHG is smaller than some input value CHGMIN, then analysis proceeds with no concern. If the change during any one time period is more than CHGMIN but less than some predetermined limit which we term CHGLIM, the calculation is repeated with the value of  $c_v$  taken as the average of the first value used (at time T) and the value at the end of the time step (at time TC) and the calculation is started all over from time T but with a suitably scaled down value for the time period.

As long as the soil is being loaded it is relatively easy to track any desired curve of  $c_v$  versus  $\sigma$ . Unloading causes problems however because we do not know, in advance, the effective stresses from which the unloading begins. Our current programs provide that when unloading occurs the coefficient of consolidation remains constant. Upon reloading,  $c_v$ , remains constant until the effective stress again reaches its maximum previous value, on the input curve of cv versus  $\sigma'$ , and then the values of  $c_v$  again track that curve.

### 5.7.1 Consolidation along predetermined e-log $\sigma^\prime$ curves

In a real field settlement problem the major interest is in settlement not in average degree of consolidation. The settlement is calculated using an equation such as:

$$S = H_0 \frac{e_0 - e}{1 + e_0}$$
 (5.58)

where  $H_0$  is the original thickness of any given layer, and  $e_0$  and e are the initial void ratio and the void ratio at the time the settlement is to be calculated, respectively. Values of  $H_0$ and  $e_0$  are input at the beginning of the analysis and values for e are obtained by linear interpolation from a plot of void ratio versus effective stress (Fig. 5.9b) using, again, the average value for the effective stress in the chosen soil layer. Some engineers prefer to work with strain diagrams. If  $\epsilon$  represents linear strain, then the settlement is given by:

$$S = \varepsilon H_0 \tag{5.59}$$

where is again obtained by linear interpolation from a plot of  $\varepsilon$  versus  $\sigma$  using the average effective stress in the layer.

Note that from the point of view of programming, it is a trivial matter to interpolate e (or  $\varepsilon$ ) versus either  $\sigma$  or log( $\sigma$ ) so our programs contain an option to interpolate on a natural or semilogarithmic scale. Execution times are, of course, longer when the semilog scale is used.

### 5.7.2 Large strains

When large fills are placed on soft clays the resulting settlements may be as much as 80% of the original thickness of the compressible layer. In such cases the normal assumption of constant H is clearly not acceptable. Since consolidation proceeds inwards from drainage boundaries, the strains vary within the compressible layer and the spacing of the nodes, which was originally uniform, now becomes nonuniform. The simple finite difference method that has previously been used, does not account for nonuniform node spacing.

One option is to formulate the program as a large strain problem. Such formulation, once adopted, must be used for all problems even though large strains are relatively rare, and thus the execution times are increased considerably with generally negligible benefit.

A simple method for partially accounting for large strain effects is the following: Maintain the number of nodes in a layer constant. At the end of each iteration determine the new layer thickness, H, and divide by (NZ-1), where NZ is the number of nodes in the layer, to obtain a new uniform spacing. Retain the same pore pressures at each node as before it was displaced. This method is obviously approximate but it is simple and does partially account for large strains.

### **5.8 Submergence Corrections**

For most embankment settlement problems, the water table is near the surface and remains at more or less constant elevation as settlement occurs. Of course, submergence of soil as it passes from above the water table to below it causes a reduction in the effective applied stress. Our computer programs were written to make a submergence correction at the end of each iteration time. The correction also includes a permanent change in water table which is sometimes brought about by the construction of a large embankment.

### **5.9** Application of Finite Difference in the Design of Wick Installations

Finite differences may be used to solve the free-strain theory for cases in which wicks or sand drains are used to accelerate consolidation. The differential equation for purely radial flow is:

$$\frac{\partial u}{\partial t} = c_r \left( \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} \right)$$
(5.60)

The following finite difference approximations are used for the partial derivatives:

$$\frac{\partial \overline{u}}{\partial t} = \frac{\overline{u}_{k,j+1} - \overline{u}_{k,j}}{\Delta t}$$
(5.61a)

$$\frac{1}{r}\frac{\partial \overline{u}}{\partial r} = \frac{\overline{u}_{k+1,j} - \overline{u}_{k-1,j}}{2r_k \Delta r}$$
(5.61b)

$$\frac{\partial^2 \overline{u}}{\partial r^2} = \frac{\overline{u}_{k-1,j} - 2\overline{u}_{k,j} + \overline{u}_{k+1,j}}{\Delta r^2}$$
(7.61c)

#### 5.9.1 Explicit solution, instantaneous loading, constant properties, linearly elastic soil

Equations 5.61a-5.61c are inserted into Eq. 5.60, which is solved for  $u_{k,j+1}$  to obtain:

$$u_{k,j+1} = u_{k-1,j}(\alpha) \left(1 - \frac{\Delta r}{2r_k}\right) + u_{k,j}(1 - 2\alpha) + u_{k+1,j}(\alpha) \left(1 + \frac{\Delta r}{2r_k}\right)$$
(5.62)

where:

$$\alpha = c_R \Delta t / \Delta r^2 \tag{5.63}$$

The factor  $\alpha$  is again limited to the range of 0 to 0.50. Since all terms on the right side of Eq. 5.62 are known, the left side can be evaluated for all values of the radius index, k, and solutions can thus be obtained for all values of time. The excess pore water pressure is set at zero at the boundary of the well for all times greater than zero, and the outer boundary is made impervious by using an imaginary node on the outside with the same pore pressure as the first node on the inside of the outer boundary.

For a linearly elastic soil, the average settlement is conveniently determined using the average degree of consolidation, U, defined as:

$$U = \frac{r_{e} \qquad r_{e}}{\int 2\pi r u_{o} dr - \int 2\pi r u dr}$$

$$U = \frac{r_{w} \qquad r_{w}}{r_{e}}$$

$$\int 2\pi r u_{o} dr$$

$$r_{w}$$
(5.64)

where  $u_0$  is used here to indicate the initial uniform excess pore water pressure. The integration is conveniently performed by assuming the isochrones are parabolic through any three successive nodes, determining the coefficients for the parabolic isochrone from the

known pore pressure at the three nodes, finding the volume under the parabola using the calculus, and then summing over all nodes, i.e., by rederiving an intergration rule similar to Simpson's 1/3 rule. The equation for the volume under the pore pressure isochrone becomes:

$$\int_{r_w}^{r_e} 2\pi r \overline{u} dr = \frac{2}{3} \pi \Delta r (r_1 u_1 + 4r_2 u_2 + 2r_3 u_3 + 4r_4 u_4 + \dots + 2r_{N-2} - 2u_{N-2} + 4r_{N-1} u_{N-1} + r_N u_N)$$
(5.65)

In Eq. 5.65, N must be odd and greater than one. A trapezoidal integration scheme can also be developed but we use Eq. 5.65.

The curves of  $T_r$  versus  $U_{r_s}$  obtained using the explicit finite difference method, are compared with curves obtained using the free strain sand drain theory in Fig. 5.10. The effect of number of radial nodes on the accuracy of the solution for N=10 is shown in Fig. 5.11.

#### 5.9.2 Crank-Nicholson method, radial flow only, constant properties

A Crank-Nicholson solution for the excess pore water pressures can be developed using the equations and techniques presented previously. The equation for the excess pore water pressures is:

$$a_1u_{k-1,j+1} + a_2u_{k,j+1} + a_3u_{k+1,j+1} = C_k$$
(5.66a)

where:

$$C_{k} = b_{1}u_{k-1,j} + b_{2}u_{k,j} + b_{3}u_{k+1,j}$$
(5.66b)

and:

$$a_1 = \alpha \left( \frac{\Delta R}{2R} - 1 \right) \tag{5.66c}$$

$$a_2 = 2(1+\alpha)$$
 (5.66d)

$$a_3 = -\alpha \left(\frac{\Delta R}{2R} + 1\right) \tag{5.66e}$$

$$b_1 = \alpha \left( 1 - \frac{\Delta R}{2R} \right) \tag{7.66f}$$

$$b_2 = 2(1-\alpha)$$
 (5.66g)

$$b_3 = \alpha \left(\frac{\Delta R}{2R} + 1\right) \tag{5.66h}$$

The solution proceeds in the manner discussed previously and thus need not be considered again.

# **5.9.3** Crank-Nicholson solution for combined vertical and radial flow and for a horizontally stratified soil system

The extension of the Crank-Nicholson method to include the more general case of a soil with combined vertical and radial flow and with horizontal stratification, with adjustments for time dependent loading, variable coefficients of vertical and radial consolidation, with non-linear stress-strain curves and adjustments for large strains, is a relatively simple matter so far as setting up the equations is concerned. The difficulty lies in writing a computer program. The differential equation to be satisfied is:

$$\frac{\partial u}{\partial t} = c_{v} \frac{\partial^{2} u}{\partial z^{2}} + c_{R} \left( \frac{1}{R} \frac{\partial u}{\partial R} + \frac{\partial^{2} u}{\partial R^{2}} \right) + \frac{dq}{dt}$$
(5.67)

We will use i to indicate the depth coordinate, k to indicate the radial coordinate, and j to indicate time.

For a node located wholely within a layer, the finite difference equation becomes:

$$u_{i-1,k,j+1} + A_k u_{i,k-1,j+1} + B_L u_{i,k,j+1} + C_k u_{i,k+1,j+1} + u_{i+1,k,j+1} = -u_{i-1,k,j} - A_k u_{i,k-1,j} + B_R u_{i,k,j} - C_k u_{i,k+1,j} - u_{i+1,k,j} + (q_{i+1}-q_i)$$
(5.68a)

where:

$$A_{k} = -\frac{\alpha_{R}}{\alpha_{z}} \left( \frac{\Delta R}{2R} - 1 \right)$$
(5.68b)

$$B_{\rm L} = -\frac{2}{\alpha_z} (1 + \alpha_z + \alpha_{\rm R}) \tag{5.68c}$$

$$C_{k} = \frac{\alpha_{R}}{\alpha_{z}} \left( \frac{\Delta R}{2R} + 1 \right)$$
(5.68d)

$$B_{\rm R} = -\frac{2}{\alpha_z} (1 - \alpha_z - \alpha_{\rm R}) \tag{5.68e}$$

and:

$$z = c_V t/(z)^2$$
 (5.68f)

$$\alpha_{\rm R} = c_{\rm R} \Delta t / (\Delta R)^2 \tag{5.68g}$$

For a node located at a horizontal interface between two layers:

$$\overline{u}_{i-1,k,j+1} + \rho_A \overline{u}_{i,k-1,j+1} + \rho_{BL} \overline{u}_{i,k,j+1} + \rho_C \overline{u}_{i,k+1,j+1} + \gamma \overline{u}_{i+1,k,j+1} = -\overline{u}_{i-1,k,j} - \rho_A \overline{u}_{i,k-1,j} + \rho_{BR} \overline{u}_{i,k,j} - \rho_C \overline{u}_{i,k+1,j} - \gamma \overline{u}_{i+1,k,j}$$
(5.69)

where:

$$\rho_{\rm A} = 1/2(A_{\rm kv} + \gamma A_{\rm kw})$$
 (5.70a)

$$\rho_{\rm BL} = 1/2(B_{\rm Lv} + \gamma B_{\rm Lw}) \tag{5.70b}$$

$$\rho_{\rm BR} = 1/2(B_{\rm Rv} + \gamma B_{\rm Rw}) \tag{5.70c}$$

$$\rho_{\rm C} = 1/2(C_{\rm kv} + \gamma C_{\rm kw}) \tag{5.70d}$$

The subscripts v and w indicate the layer above and below the interface, respectively and the coefficients  $A_k$ ,  $B_L$ ,  $B_R$ , and  $C_k$  are defined in Eqs. 5.68a-5.68d.

Equations 5.68 and 5.69 are modified further for nodes located at either freely draining or impervious vertical or horizontal boundaries. By the time all possible combinations of freely draining and impervious boundaries are taken into account the programmer ends up with about 21 equations plus equations for the coefficients.

### **5.10 Other Cases**

These notes have demonstrated the general method of application of finite difference techniques to the solution of consolidation problems. It is evident that different schemes could be used and that additional problems can be analyzed. Finite differences can easily be used to determine two- and three-dimensional dissipation of pore water pressures, as under footings, but determination or resulting earth movements is not so simple. Finite differences may be used to account for secondary effects both during and after primary consolidation, and for effects of partial saturation. Since these notes are intended only to introduce the method and demonstrate its use, no effort will be made to cover additional cases.