UNIT 2

ANALYSIS OF TIME RATES OF PRIMARY CONSOLIDATION FOR THE SIMPLEST CASE

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2.1 Introduction

In the context of this set of notes, the "simplest case" involves the following requirements:

1. deformation occurs only in the vertical direction,
2. water flow occurs in only the vertical direction,
3. water flows in accord with Darcy's law,
4. strains are small,
5. all soil properties are constant,
6. the soil is linearly elastic,
7. the soil is homogeneous,
8. the only resistance to volume change is hydrodynamic,
9. the soil is saturated,
10. the pore water and soil grains are incompressible,
11. the soil is loaded instantaneously at time zero and the load does not vary with time,
12. the two horizontal boundaries are either freely draining or impervious.

There are no known cases in the real world where these assumptions are all satisfied but the resulting theory is commonly used in engineering practice because of its simplicity. From our point of view, this simplified theory is a logical place to begin the development of the more general solutions which will be examined later.

In this set of notes we will derive the governing differential equation and solve it for the case listed above.

2.2 Derivation of Differential Equation

The derivation of the differential equation governing this problem utilizes a differential element of soil as shown in Fig. 2.1. The z axis is directed positively downwards. The area of element perpendicular to the z axis is dA. The time rate of decrease in volume, dV, is the difference between the rate of outflow, q_{out}, and inflow, q_{in}:

$$-\frac{\partial dV}{\partial t} = q_{out} - q_{in} = (\frac{\partial q}{\partial z})dz \tag{2.1}$$

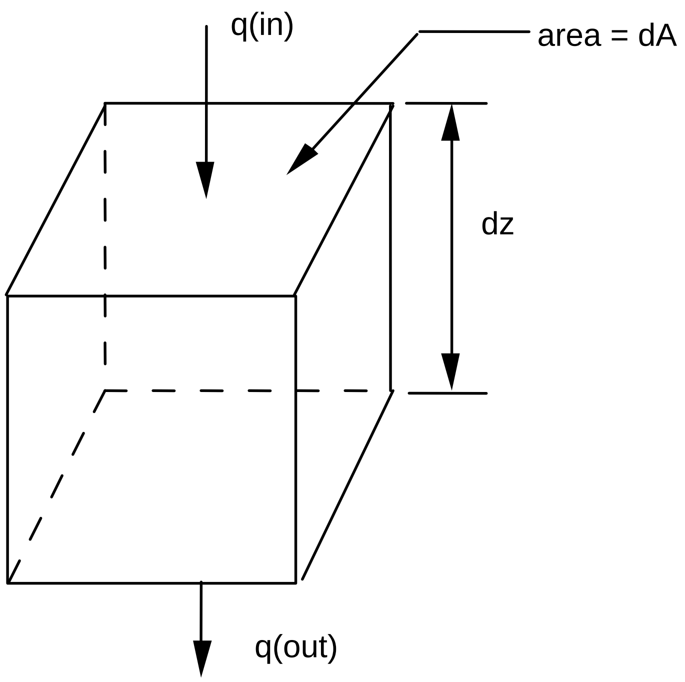

Fig. 2.1 Differential Element used in Developing the Equations for One-Dimensional Consolidation

The flow rate, q, is given by Darcy's law (assumption 3), which can be written in terms of excess pore water pressures as:

$$q = -\frac{k}{\gamma_w}\frac{\partial \bar{u}}{\partial z} dA \tag{2.2}$$

where γ_W is the unit weight of water and $\bar{u}$ is the excess pore water pressure given by:

$$\bar{u} = u - u_s \tag{2.3}$$

where u is the total pore water pressure and u_S is the static pore water pressure given by:

$$u_s = \gamma_w (z - z_{WT}) \tag{2.4}$$

where z is the depth to the element measured from the ground surface and z_{WT} is the depth to the water table.

Equation 2.2 is inserted into Eq. 2.1:

$$-\frac{\partial dV}{\partial t} = \frac{\partial}{\partial z}\left(-\frac{k}{\gamma_w}\frac{\partial \bar{u}}{\partial z} dA\right) dz \tag{2.5}$$

We assume k is independent of z (assumption 7) and replace dA dz with dV:

$$\frac{\partial dV}{\partial t} = \frac{k}{\gamma_w}\frac{\partial^2 \overline{u}}{\partial z^2} dV \qquad (2.6)$$

The decrease in volume occurs only by expulsion of pore water (voids):

$$\frac{\partial (dV)}{\partial t} = -\frac{\partial (dV_V)}{\partial t} \qquad (2.7)$$

with dV_V the volume of voids and the minus sign is used to satisfy our sign convention (dV is positive for a decrease in volume). We will choose to use void ratios (e) at this point and note that:

$$dV_V = \frac{e}{1+e} \quad dV \qquad (2.8)$$

and also that the volume of solids, dV_S, is:

$$dV_S = \frac{1}{1+e} \quad dV \qquad (2.9)$$

No solids enter or leave the element so dV_S is independent of time. Thus:

$$\frac{\partial (dV)}{\partial t} = -\frac{\partial (\frac{e}{1+e}\ dV)}{\partial t} = -\frac{\partial (e\ dV_S)}{\partial t} = -(\frac{\partial e}{\partial t})dV_S$$

$$= -(\frac{\partial e}{\partial t})\ \frac{dV}{1+e} \qquad (2.10)$$

Equations 6 and 10 are equated, and dV canceled:

$$\frac{k(1+e)}{\gamma_w}\frac{\partial^2 \overline{u}}{\partial z^2} = \frac{\partial e}{\partial t} \qquad (2.11)$$

We need to reduce the dependent variables (u and e) to a single variable, and we choose u. First, define:

$$a_v = -\frac{de}{d\overline{\sigma}} \qquad (2.12)$$

where a_V is the **coefficient of compressibility**, and note that a_V is the slope of a stress (σ)-strain (Δe) curve. Thus, a constant a_V (assumption 5) means that the soil is linearly elastic (assumption 6). From the definition of effective stress:

$$\overline{\sigma} = \sigma - u \qquad (2.13)$$

it follows that:

$$d\overline{\sigma} = d\sigma - du \qquad (2.14)$$

Thus:

$$\frac{\partial e}{\partial t} = -a_v \frac{\partial \overline{\sigma}}{\partial t} = -a_v \left(\frac{\partial \sigma}{\partial t} - \frac{\partial u}{\partial t} \right) \qquad (2.15)$$

We assume that the total stress is independent of time (assumption 11) so $\frac{\partial \sigma}{\partial t} = 0$. Equation 2.15, with $\frac{\partial \sigma}{\partial t}$ removed, is inserted into Eq. 2.11:

$$\frac{\partial \overline{u}}{\partial t} = \frac{k(1+e)}{a_v \gamma_w} \frac{\partial^2 \overline{u}}{\partial z^2} \qquad (2.16)$$

Equation 2.16 is the desired differential equation. However, to simplify writing we use:

$$\frac{\partial \overline{u}}{\partial t} = c_v \frac{\partial^2 \overline{u}}{\partial z^2} \qquad (2.17)$$

where c_V is termed the **coefficient of consolidation**:

$$c_v = \frac{k(1+e)}{a_v \gamma_w} \qquad (2.18)$$

Equation 17 is called Terzaghi's differential equation (Terzaghi, 1923, 1925).

Note that we could have solved for e or σ instead of u, and could have used linear strain (ε) in place of void ratio.

2.3 Solution of the Differential Equation of Consolidation

Equation 2.17 is a common equation in science and engineering. It governs temperature (T in place of u) in bodies undergoing non-steady heat flow (c_V becomes the coefficient of thermal diffusivity). It governs the flow of electricity in conductors, diffusion of solutes into solvents, and other phenomena. The solution is found in essentially all texts on partial differential equations, and many texts in differential equations and advanced calculus.

Several types of solutions can be derived which have different appearances but yield the same answers. The solution generally used is obtained by assuming that the pore water pressure, u(z,t) can be expressed as the multiple of two functions, one involving only depth and one involving only time:

$$\overline{u}(z,t) = F(z)G(t) \qquad (2.19)$$

This assumption seems like a reasonable one because the governing differential equations involve partial derivatives, implying a separate variation of u with respect to depth and time and making it unlikely that there is a coupling term (one involving both z and t). To save on space we will use $\bar{u}$ in place of the more generally used $\bar{u}(z,t)$.

Equation 2.19 is inserted into Eq. 2.17 and the terms factored to obtain:

$$\frac{F''(z)}{F(z)} = \frac{G'(t)}{c_v G(t)} \tag{2.20}$$

where the single and double primes indicate the first and second derivatives of the function with respect to the variable inside the parenthesis.

The left side of Eq. 2.20 is a function only of position and the right side is a function only of time. Because the two are equal it follows that neither can depend on either position or time. Thus, solutions can be obtained by setting the sides equal to a constant, solving the resulting differential equations for the two functions and inserting them into Eq. 2.19. The constant is taken, for later convenience, as $-A^2$.

The left side of Eq. 2.20 is set equal to $-A^2$ and is solved to obtain:

$$F(z) = C_1\cos(Az) + C_2\sin(Az) \tag{2.21}$$

Solution of the right side of Eq. 2.20 yields:

$$G(t) = C_3\exp(-A^2c_vt) \tag{2.22}$$

Equations 2.21 and 2.22 are then inserted into Eq. 2.19 to obtain:

$$\bar{u} = (C_4\cos Az + C_5\sin Az)\exp(-A^2c_vt) \tag{2.23}$$

Equation 2.23 is a general solution to Eq. 2.17 (remember that Eq. 2.17 is valid only for conditions satisfying the assumptions listed earlier). We must now define the conditions of a particular problem which we wish to solve. We will choose to analyze pore water pressures in a single layer, of thickness 2H, with top and bottom boundaries freely draining. We will define the z axis as positive downwards starting at the top surface. Thus, there are two boundary conditions:

$$\bar{u}(0,t) = 0 \tag{2.24a}$$

$$\bar{u}(2H,t) = 0 \tag{2.24b}$$

where (0,t) means at depth zero and at all times, and (2H,t) means depth 2H and all t. In addition, we need to know the initial variation of excess pore water pressure with respect to depth, i.e., the initial condition. For an areal fill applied instantly to a saturated soil, the initial excess pore water pressure, $\bar{u}(z,0)$, is independent of depth and equals the applied stress (Appendix A):

$$\bar{u}(z,0) = \bar{u}_i \qquad (2.24c)$$

We now have three conditions and three unknowns.

The first boundary condition is satisfied if C_4 is zero. Thus, Eq. 2.23 reduces to:

$$\bar{u} = (C_5 \sin Az) \exp(-A^2 c_v t) \qquad (2.25)$$

The second boundary condition is satisfied if sin(Az) is zero, meaning that the argument can have values o, ±π, ±2π, ±3π, ... when z = 2H. We will not take up space to prove it, but a valid solution can be obtained using all values of n from -∞ to +∞ or using just positive values. We will simplify the analysis by using only positive values. Thus:

$$A\,2H = n\pi$$

or:

$$A = \frac{n\pi}{2H} \qquad (2.26)$$

Equation 2.26 is inserted into Eq. 2.25 to obtain:

$$\bar{u} = C_5 \sin\left(\frac{n\pi z}{2H}\right)\exp\left(\frac{-n^2\pi^2 c_v t}{4H^2}\right) \qquad (2.27)$$

To simplify writing we choose to define a dimensionless term T as:

$$T = \frac{c_v t}{H^2} \qquad (2.28)$$

and call T the **time factor**. Thus:

$$\bar{u} = C_5 \sin\left(\frac{npz}{2H}\right)\exp\left(\frac{-n^2\pi^2 T}{4}\right) \qquad (2.29)$$

If Eq. 2.29 yields $\bar{u}$ (2H,t) = 0 for each and every value of n, then we can sum terms for all values of n and still satisfy the boundary condition (the sum of terms, each of which is zero, is also zero):

$$\bar{u} = \sum_{n=0}^{\infty} C_n \sin\left(\frac{n\pi z}{2H}\right)\exp\left(\frac{-n^2\pi^2 T}{4}\right) \qquad (2.30)$$

We have thus preserved all possible terms (we cannot arbitrarily drop some terms and then expect to get the correct solution). At this time we have no idea of what value, or values, C_5 will have so we write it as C_n on the assumption that it may depend on n.

The initial condition now requires that $\bar{u} = \bar{u}_i$ if we insert T = 0 into Equation 30:

$$\bar{u}_i = \sum_{n=0}^{\infty} C_n \sin\left(\frac{n\pi z}{2H}\right) \qquad (2.31)$$

Equation 2.31 is recognized as a Fourier sine series (Appendix B).

For the simple problem under consideration the Fourier coefficients are given by:

$$C_n = \frac{1}{H}\int_0^{2H} \bar{u}_i \sin\left(\frac{n\pi z}{2H}\right) dz \qquad (2.32)$$

Equation 2.32 is substituted into Eq. 2.30 to obtain:

$$\bar{u} = \sum_{n=0}^{\infty}\left[\frac{1}{H}\int_0^{2H} \bar{u}_i \sin\left(\frac{n\pi z}{2H}\right) dz\right] \sin\left(\frac{n\pi z}{2H}\right) \exp\left(-\frac{n^2\pi^2}{4}T\right) \qquad (2.33)$$

Integration yields:

$$\bar{u} = \sum_{n=0}^{\infty} \frac{2\bar{u}_i}{n\pi}\left[1-\cos(n\pi)\right] \sin\left(\frac{n\pi z}{2H}\right) \exp\left(-\frac{n^2\pi^2}{4}T\right) \qquad (2.34)$$

The term (1 - cos nπ) has a value of two when n is odd and zero when n is even. Thus, the equation exists only for odd values of n. Let n be replaced by another integer m such that all values of m between zero and infinity yield solutions, i.e., let n = 2m + 1 where m = 0, 1, 2 . . . .

$$M = \frac{\pi}{2}(2m+1) = \frac{n\pi}{2} \qquad (2.35)$$

Substitution of Eq. 2.35 into Eq. 2.34 yields:

$$\bar{u} = \sum_{m=0}^{\infty} \frac{2u_i}{M} \sin\left(\frac{Mz}{H}\right) \exp(-M^2T) \qquad (2.36)$$

Equation 2.36 is the desired solution.

Equation 36 can be solved to obtain the distribution of the excess pore water pressure as a function of depth for various values of the time factor but a more useful solution is obtained in terms of the average degree of consolidation, which is considered next.

Degree of Consolidation

It is useful to define a dimensionless parameter to represent the fraction of the ultimate consolidation that has been completed. If a linear relationship between void ratio and effective stress is assumed for the range under consideration, then the **degree of consolidation**, U(z,t), at the depth and time under consideration can be defined as:

$$U(z,t) = \frac{\bar{\sigma} - \bar{\sigma}_i}{\bar{\sigma}_f - \bar{\sigma}_i} = \frac{\bar{u}_i - \bar{u}}{\bar{u}_i} = \frac{e_i - e}{e_i - e_f} \quad (2.37)$$

where the subscripts i and f indicate initial and final conditions and the lack of a subscript indicates an intermediate condition. The differential equation of consolidation was solved in terms of excess pore water pressures; thus, it is convenient to use excess pore water pressures in evaluating the degree of consolidation.

Substitution of Eq. 2.36 into Eq. 2.37 then yields:

$$U(z,t) = 1 - \sum_{m=0}^{\infty} \left(\frac{2}{M}\right) \sin\left(\frac{Mz}{H}\right) \exp(-M^2T) \quad (2.38)$$

In engineering practice it is usually of interest to know the average degree of consolidation for the entire stratum. The **average degree of consolidation**, U, is determined by integration:

$$U = \frac{\int_0^{2H} \bar{u}_i dz - \int_0^{2H} \bar{u} dz}{\int_0^{2H} \bar{u}_i dz} \quad (2.39)$$

Substitution of Eq. 2.36 into Eq. 2.39 and integration yields:

$$U = 1 - \sum_{m=0}^{\infty} \frac{2}{M^2} \exp(-M^2T) \quad (2.40)$$

Because of the fact that U is a single valued function of T, Equation 2.40 need be solved only once and the results tabulated for general reference. The numerical relationship between T and U is presented in Table 2.1.

Fox (1948) used LaPlace transformations to demonstrate that an accurate approximation of Eq. 2.40 for small values of T would have the form:

$$T = \frac{\pi}{4} U^2 \quad (2.41)$$

Table 2.1 - Time Factors for a Rectangular Stress Surface* and Double Drainage

U(t),%	T	U(t),%	T
0	0.000	55	.239
5	.002	60	.286
10	.008	65	.340
15	.018	70	.403
20	.031	75	.477
25	.049	80	.567
30	.071	85	.684
35	.096	90	.848
40	.126	95	1.129
45	.159	99	1.781
50	.197	100	∞

* Terzaghi (Terzaghi and Frohlich, 1936) used the word lastflache to denote the initial distribution of excess pore water pressure. I have translated that word as stress surface.

He demonstrated that the difference between U calculated using Eqs. 2.40 and 2.41 was only 0.0002% for T = 0.1 and 0.82% for T = 0.3 (U = 60%). Thus, Eq. 2.41 can be used to calculate the T-U relationship for all values of U less than 60% with an error of less than one percent.

Evaluation of Eq. 2.40 will demonstrate that the series converges rapidly for large values of T (only one or two terms are needed for high values of T) but that the number of terms required to achieve some specified accuracy increases rapidly as T decreases. Thus, Eq. 2.41 is useful at small values of T and Eq. 2.40 is useful for the larger values, say T greater than 0.3.

Isochrones

The degree to which consolidation has taken place within a clay layer at various times is conveniently depicted using curves of u/u_i versus z/2H for various values of time. Such curves are termed isochrones. Isochrones for the rectangular stress surface and double drainage are shown in Figure 2.2.

The isochrones are symmetrical about the mid-depth of the doubly drained layer. Hence, the hydraulic gradient is zero at the mid-depth for all degrees of consolidation and no water flows across this plane during any stage of consolidation. An impervious membrane could be inserted in the layer at this depth without influencing the progress of consolidation. Thus, the theory of consolidation, as derived in this chapter for a rectangular stress surface, can be applied directly to a singly drained layer with the definition that the thickness of the singly drained layer is H, not 2H. If desired, the consolidation equations could be rederived with the second boundary condition replaced by $\partial u(H,t)/\partial z = 0$. The revised derivation leads to exactly the same equations as those presented previously for the doubly drained layer of thickness 2H.

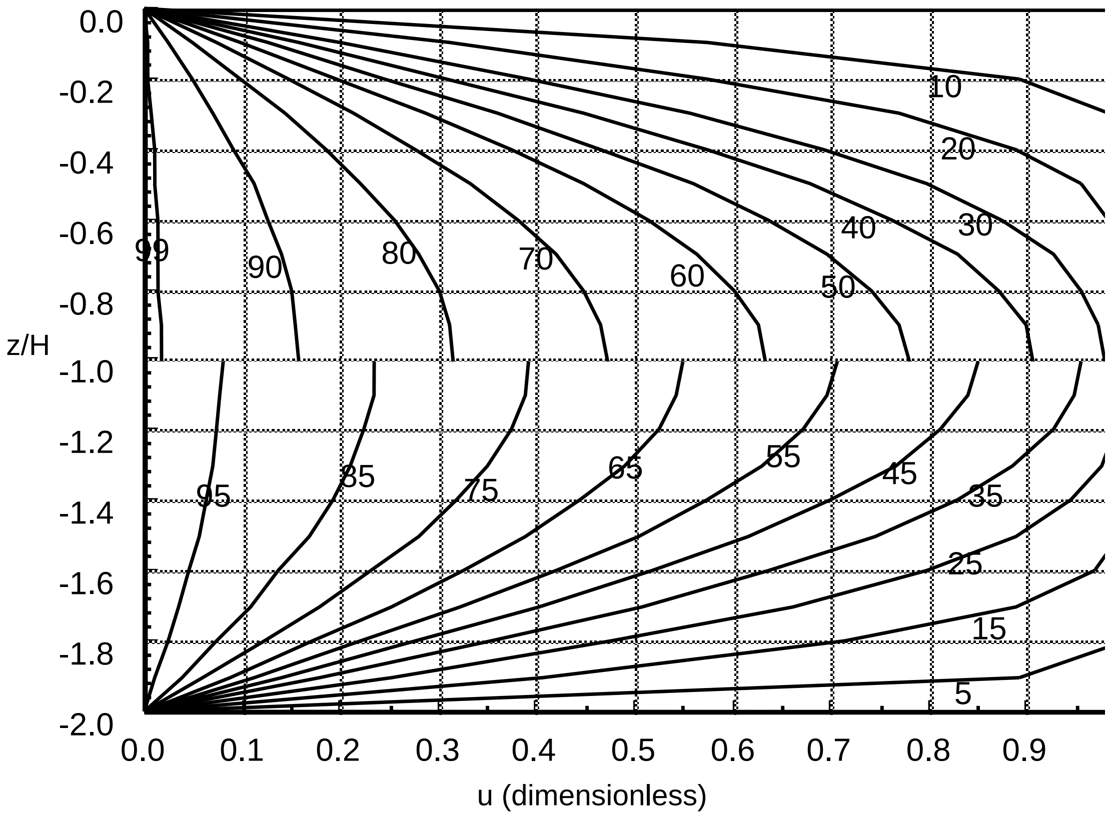

Fig. 2.2 Isochrones for Rectangular Stress Surface, Double Drainage

Time Rate of Settlement

The change of void ratio, Δe, of a differential element of soil within a consolidating layer can be expressed in terms of either the change in thickness of the differential element, dS, or the change in pore water pressure, Δu. Thus:

$$\int dS = -\int \frac{a_V \Delta u}{1+e}\, dz \tag{2.42}$$

As for previous analyses, the void ratio and coefficient of compressibility are assumed to be independent of depth. Equation 42 is integrated to obtain the settlement at any time t:

$$S = \frac{a_v}{1+e}\int_0^{2H} (\bar{u}_i - \bar{u})dz \tag{2.43}$$

Apparently, the ultimate settlement is:

$$S_u = \frac{a_v}{1+e}\int_0^{2H} \bar{u}_i dz \tag{2.44}$$

If Eq. 2.43 is divided by Eq. 2.44, the right side is seen to be exactly equal to the average degree of consolidation (Eq. 2.39). Thus:

$$S = S_u U \qquad (2.45)$$

As an example of the calculation of the time-settlement curve, consideration is again given to the case of the 30-foot thick doubly drained clay layer overlain by ten feet of sand and loaded with ten feet of compacted fill. The calculated ultimate settlement was 21.2 inches. The coefficient of consolidation of the clay is 5x10-4 cm2/sec (4.65x10-2 ft2/day). Equation 28 is rearranged to yield:

$$t(\text{days}) = \frac{TH^2}{c_v} = \frac{15^2}{0.0465} \quad T = 4840T$$

When consolidation is 50-percent completed the settlement is (0.50) (21.2) = 10.6 inches and the time is (4840) (0.197) = 955 days. Other points are obtained in a similar manner.

2.4 References

Skempton, A.W. (1954), "The Pore-Pressure Coefficients A and B," *Geotechnique*, Vol. 4, pp. 143-147.

Terzaghi, K. T. (1923), "Die Berechnung der Durchlassigkeitsziffer des Tones aus dem Verlauf der Hydrodynamischen Spannungscheinungen," Sitz. Akad. Wissen, Wein Math-natur. Kl. Abt. 11a, Bd. 132, pp. 105-124.

Terzaghi, K. T. (1925), *Erdbaumechanik auf Bodenphysikalischer Grundlage*, Franz Deuticke, Vienna, 399 pp.

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Appendix A

DERIVATION OF EQUATION FOR INITIAL EXCESS PORE WATER PRESSURE

A differential element of soil is considered which is subjected to an instantaneous increase in vertical total stress under conditions of zero lateral strain and zero lateral drainage. The element consists of pore water, and a skeleton of soil particles which transmits the effective stress. When the element is subjected to an increase in total stress, a volume change must occur, even though there has been no time for drainage, because all materials have finite compressibilities. The solid mineral matter is so much less compressible than the pore water, however, that compression of the mineral matter can be ignored. Hence, any changes in volume of the element, at the instant of loading and before any drainage can occur, must result from a change in the volume of the pore water.

The compressibility of any substance may be defined using the equation:

$$C = -\frac{dV/V}{dp} \qquad \text{(A.1)}$$

where dV is the change in volume resulting from the application of a pressure increment dp to a volume V, and C is the coefficient of compressibility. If the total volume of the differential element of saturated soil is V, then the volume of pore water is $\frac{e}{1+e}V$, and the change in volume of water, dV_W, resulting from a change in pore water pressure, du, must be:

$$dV_W = -C_W\left(\frac{e}{1+e}V\right)du \qquad \text{(A.2)}$$

where C_W is the coefficient of compressibility of the water.

The change in volume of the element of soil represents not only a compression of the pore water but also a compression of the soil skeleton. The compressibility of the soil skeleton is defined using the equation:

$$a_v = -de/d\overline{\sigma} \qquad \text{(A.3)}$$

The change in volume of the soil skeleton is then:

$$dV = \frac{a_v}{1+e}Vd\overline{\sigma} \qquad \text{(A.4)}$$

The change in volume of the soil skeleton is equal to the total change in volume of the element which is equal to the change in volume of the water. Thus Eqs. A.2 and A.4 may be equated and rearranged to yield:

$$d\overline{\sigma} = \frac{c_w e}{a_v} du \tag{A.5}$$

Terzaghi's effective-stress equation may be written:

$$d\overline{\sigma} = d\sigma - du \tag{A.6}$$

Equation A.6 is substituted in Equation A.5 to obtain:

$$\frac{du}{d\overline{\sigma}} = \frac{1}{1 + \frac{c_w e}{a_v}} \tag{A.7}$$

Equation A.7 is of the same form as that derived by Skempton (1954) and yields exactly the same conclusions. The coefficient of compressibility of water is about 3.4 x 10-6 psi-1. Insertion of typical values for the void ratio and coefficient of compressibility of the soil lead to a value of du/ds of about 0.999. Thus, under the loading conditions assumed here, essentially all of the applied stress is taken by the pore water.

The foregoing statement is restricted to the case of saturated soil of normal compressibility subjected to a uniform load under conditions of no lateral strain.

Appendix B

EVALUATION OF THE COEFFICIENTS IN A FOURIER SERIES

Introduction

Fourier series are discussed in textbooks on advanced calculus. Such books should be consulted for a more thorough discussion of the topic. For the solution of problems involving one dimensional consolidation of saturated soils, following Terzaghi's assumptions, the discussion can be simplified greatly. In this set of notes the attempt is made to provide a sufficient amount of information so that one can solve one dimensional consolidation problems but no more. The mathematics is presented in logical order but largely without explanation.

Conversions Between Trigonometric and Exponential Forms

Let z represent a complex variable given by:

$$z = x + iy \qquad \text{(B.1)}$$

Exponential and trigonometric forms of the complex variable can be evaluated using infinite series as follows:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \qquad \text{(B.2)}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \qquad \text{(B.3)}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \qquad \text{(B.4)}$$

A comparison of the series demonstrates that:

$$e^{iz} = \cos z + i \sin z \qquad \text{(B.5)}$$

and:

$$e^{-iz} = \cos z - i \sin z \qquad \text{(B.6)}$$

Simultaneous solution of Equations B.5 and B.6 yields:

$$\sin z = \frac{1}{2i} (e^{iz} - e^{-iz}) \qquad \text{(B.7)}$$

and:

$$\cos z = \frac{1}{2} (e^{iz} + e^{-iz}) \qquad \text{(B.8)}$$

Even and Odd Functions

In the discussion to follow it will be convenient to use the terms "even" and "odd" in describing functions. If the independent variable is x and f(x) is some function of x, then the function is described as even if:

$$f(-x) = f(x) \qquad (B.9)$$

and odd if:

$$f(-x) = -f(x) \qquad (B.10)$$

Substitution will demonstrate that cos(x) and x^n, where n is an even integer, are even junctions and sin x and x^n, where n is an odd integer, are odd functions.

A function, g(x), may be neither odd nor even, e.g., g(x) might be 1 + sin x + cos x. It may be convenient to divide g(x) into an even function and an odd function. This may accomplished as follows:

$$g(x) = \frac{1}{2}\{g(x) + g(-x)\} + \frac{1}{2}\{g(x) - g(-x)\} \qquad (B.11)$$

The first term on the right side is apparently even and the second is odd.

In some consolidation problems one encounters functions that are "even harmonic" or " odd harmonic." If the function has a period of p, and L = 0.5p, then an even harmonic function is one satisfying:

$$f(x + L) = f(x) \qquad (B.12)$$

and an odd harmonic function satisfies:

$$f(x + L) = -f(x) \qquad (B.13)$$

Periods of Functions

If a function f(x) is periodic, then the period p is given by:

$$f(x + p) = f(x) \qquad (B.14)$$

Further, if a function has a period p it also has a period np where n is an integer ranging from minus infinity to plus infinity.

In developing equations for the evaluation of the coefficients in Fourier series, it is convenient to integrate functions to find average values. In this respect, the following equations are taken as self evident:

$$\int_a^{a+p} f(x)\,dx = \int_0^{p} f(x)\,dx = \frac{1}{n}\int_a^{a+np} f(x)\,dx = \frac{1}{n}\int_0^{np} f(x)dx \qquad (B.15)$$

where "a" is any value of x and p is the period.

In performing integrations involving exponential, sine, and cosine terms, it is necessary to determine the periods of the functions. The period of the function sin(k x) is found as follows:

$$\sin(k\omega x) = \sin[k\omega(x+p)] = \sin(k\omega x + 2\pi)$$

$$k\omega x + k\omega p = k\omega x + 2\pi$$

$$p = \frac{2\pi}{k\omega} \qquad \text{(B.16)}$$

The period of cos(kωx) is also given by Eq. B.16. Further, an exponential function exp (ikωx) may be expressed in terms of sines and cosines using Eqs. B.5 and B.6. Since the sines and cosines have the same peroid, by Eqs. B.5 and B.6 the period of the exponential function is also given by Eq. B.16.

Definition of a Fourier Series

The Fourier series is given by the equation:

$$f(x) = a_o + a_1\cos(\omega x) + b_1\sin(1\omega x) + a_2\cos(2\omega x) + b_2\sin(2\omega x) +$$

$$= a_o + \sum_{n=1}^{\infty}[a_n\cos(n\omega x) + b_n\sin(n\omega x)] \qquad \text{(B.17)}$$

Average Values of Functions

When Equation B.17 is encountered in consolidation problems, f(x) is a known function and the problem is to determine the values of the coefficients a_o, a_n, and b_n. They can be evaluated by a straight-forward process of integration but it conserves work and time to develop a set of rules first and then apply these rules to the evaluation of the coefficients. The rules follow:

Rule 1 - We wish to determine the average value, f(x), of sin(kωx) and cos (kωx) over one period p. The average of the sine is:

$$f(x) = \frac{1}{p}\int_0^p \sin(k\omega x)dx \quad = -\frac{1}{pkw}\cos(k\omega x)\Big|_0^p \quad = 0$$

Similarly, the average of the cosine function is found to be zero. These average values could, of course, have been written down just by inspection of the shape of the sine or cosine function. Thus, rule one becomes:

For kω > 0, and p = 2π/kω, the average value of cos(kωx) or sin (kωx) for the range in x from 0 to p, is zero.

Rule 2 - We now wish to determine the average value of a function that is either the sine or cosine of kωx times either the sine or cosine of nωx, where k and n are 1, 2, 3, . . . The common period is 2π/ω. As an example of the calculation, use sin (kωx)cos(nωx):

$$\sin(k\omega x) = \frac{1}{2i} \quad [eik\omega x - e\text{-}ik\omega x]$$

$$\cos(n\omega x) = 1/2[ein\omega x + e\text{-}in\omega x]$$

$$\sin(k\omega x)\cos(n\omega x) = \frac{1}{4\,i} \quad [e^{i\omega x(n+k)} - e^{i\omega x(n-k)} - e^{-i\omega x(n+k)} + e^{-i\omega x(n-k)}]$$

Let 1=n+k and m=n-k

$$\sin(k\omega x)\cos(n\omega x) = \frac{1}{4\,i} \quad [e^{il\omega x} - e^{-il\omega x} - e^{im\omega x} + e^{-im\omega x}]$$

Substitute: $e^{il\omega x} - e^{-il\omega x} = 2i \sin(1\omega x)$

$$e^{im\omega k} - e^{-im\omega x} = \omega i \sin(m\omega x)$$

$$\sin(k\omega x)\cos(n\omega x) = 1/2 \sin(1\omega x) - 1/2 \sin(m\omega x)$$

By Rule 1 the average value of the two sine terms for a range of one period is zero. Thus the average value of sin(kωx) cos(nωx) is zero. Similar derivations may be used to evaluate the multiple of sin(kωx)sin(nωx) and cos(kωx)cos(nωx). In these cases it is found that the answer depends on whether n and k are the same or different. The results of the calculations are:

Rule 2a - **For k, n, and ω greater than zero, the average of sin(kωx) cos(nωx) or cos(kωx)sin(nωx), for the interval O, p, is zero, regardless of whether or not n and k are equal.**

Rule 2b - **For k and n greater than zero and equal, and ω greater than zero, the average of the functions $\sin^2$ (kωx) or $\cos^2$ (kωx) for the interval 0,p is 1/2.**

Rule 2c - **For k and n greater than zero and unequal, and w greater than zero, the average of sin(kωx)sin(nωx) or cos(nωx), for the interval 0,p is zero.**

Application of Rules to Fourier Series

We now return to the Fourier series, Equation B.17. The average value of the series, f(x), may be determined by term-by-term integration:

$$f(x)=\frac{1}{p}\int_{o}^{p} f(x)dx = \frac{1}{p}\int_{o}^{p} a_o dx + \frac{1}{p}\int_{o}^{p} a_1\cos(\omega x)dx+\frac{1}{p}\int_{o}^{p} b_1\sin(\omega x)dx$$

$$+\frac{1}{p}\int_{o}^{p} a_2\cos(2\omega x)dx + \ldots$$

However, the average value of the constant a_o is itself and the average values of the sine and cosine terms are all zero by Rule 1. Thus the series may be written:

$$a_o = \frac{1}{p}\int_{o}^{p} f(x)\,dx \qquad \text{(B.18)}$$

If f(x) is known, Eq. B.18 may be used to evaluate the constant a_o in the Fourier series.

Suppose, now, that Eq. B.17 is multiplied by cos(kwx) and averaged over the period p. We obtain:

$$\frac{1}{p}\int_{o}^{p} f(x)\cos(k\omega x)=\frac{1}{p}\int_{o}^{p} a_o\cos(k\omega x)dx + \frac{1}{p}\int_{o}^{p} a_1\cos(\omega x)\cos(k\omega x)dx$$

$$+\frac{1}{p}\int_{o}^{p} b_1\sin(\omega x)\cos(k\omega x)\,dx + \ldots$$

However, by Rule 1:

$$\frac{1}{p}\int_{o}^{p} a_o\cos(k\omega x)\,dx = 0$$

By Rule 2a:

$$\frac{1}{p}\int_{o}^{p} a_n\cos(n\omega x)\cos(k\omega x)\,dx = 0 \qquad n\neq k$$

By Rule 2b:

$$\frac{1}{p}\int_{o}^{p} a_n\cos(n\omega x)\cos(k\omega x)\,dx = 1/2\,a_n \qquad n=k$$

By Rule 2c:

$$\frac{1}{p}\int_{o}^{p} b_n \sin(n\omega x)\cos(k\omega x)\,dx = 0$$

Thus, among the infinite number of terms on the right side of Equation B.17, only one has a value other than zero. We solve for a_n to obtain:

$$a_n = \frac{2}{p}\int_{o}^{p} f(x)\cos n\omega x\,dx \qquad (B.19)$$

Similarly, Eq. B.17 may be multiplied by sin(n x) and averaged to obtain an equation for the coefficients of the sine terms. Again only one term is non-zero and we obtain:

$$b_n = \frac{2}{p}\int_{o}^{p} f(x)\sin n\omega x\,dx \qquad (B.20)$$

Equations B.18-B.20 are used to evaluate the coefficients in the Fourier series, Eq. B.17. The function f(x) must be known. Further, f(x) must be single valued, except at points of discontinuity, and have arcs of finite length. Waves satisfying these requirements are shown in Fig. B.1. The wave in Fig. B.1a apparently is an odd function whereas the one in Fig. B.1b is an even function.

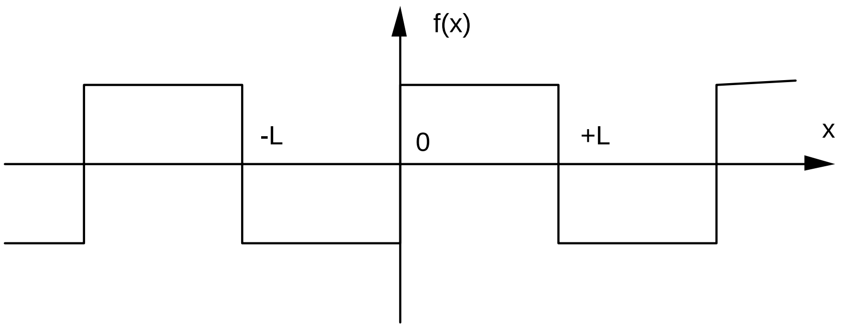

Fig. B.1A Odd Periodic Function

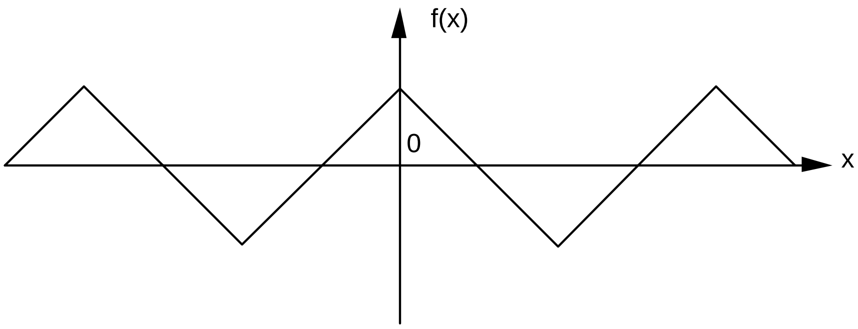

Fig. B.1B Even Periodic Function

Application to Non-Periodic Functions

The function f(x) need not actually be periodic. It may be represented by a Fourier series in some range $a \le x \le a+p$. The series representation will have values outside of this range but such values may be irrelevant to the real physical problem.

Application to Periodic Functions

In determining the values for the coefficients in a Fourier series it would be convenient if we could determine by inspection which coefficients are zero and which are finite. From Equation 18 it is apparent that the coefficient a_o must be zero if the average value of the function through one period is zero. Thus a_o is zero for the odd wave of Fig. B.1a and finite for the even wave of Fig. B.1b. In fact, it is apparent that a_o must always be zero for an odd function.

If f(x) is an even function then the multiple f(x)cos(nωx) (Equation B.19) is an even function (the multiple of two even functions is another even function) and at least one of the coefficients an will exist. Only one coefficient would exist if f(x) = cos x, and in this case it would be given by $a_1 = 1$ and $\omega = 1$. The coefficients a_n can be found by integrating Equation B.19. However, we can replace f(x) cos nwx with g(x) and note that g(x) is an even function. Thus:

$$\int_o^p f(x)\cos(n\omega x)dx = \int_{-L}^{+L} g(x)dx = \int_{-L}^{0} g(x)dx + \int_{0}^{+L} g(x)dx$$

where L is half of the period. For an even function (Equation B.9) for every value g(x) there is an equal value at -x so:

$$\int_{-L}^{o} g(x)dx = \int_{o}^{+L} g(x)dx \qquad \text{(B.21)}$$

Thus, we can simplify integration to:

$$a_n = \frac{2}{L}\int_{o}^{+L} f(x)\cos(n\omega x)dx \qquad \text{(B.22)}$$

Since sin(nωx) is an odd function, the multiple of f(x)sin(nωx) is the multiple of an even and an odd function and will therefore be an odd function, and will yield zero when integrated over one period. Thus, none of the coefficients b_n (Equation B.20) will be non-zero.

If f(x) is an odd function then f(x)cos(nωx) is odd and its integral is zero so no cosine terms exist. However, f(x)sin(nωx) is an even function so Equation B.21 is reapplied and:

$$b_n = \frac{2}{L}\int_{o}^{L} f(x)\sin(n\omega x)dx \qquad (B.23)$$

Accuracy of Fourier Series

It may be noted that the function f(x') actually defined by a Fourier series approaches the function f(x) as the number of terms in the series approaches infinity, except at points of discontinuity. At such points the function f(x') equals the average value f(x) at the point of discontinuity, where f(x) is double valued. Further, F(x') overshoots f(x) by about 18% in a zone on each side of the discontinuity. However, as the number of terms increases the thickness of the zone of overshoot decreases and a sufficient number of terms can be used to ensure that an insignificant error results form the overshoot.